Probing Quantum Field Theory at Strong Coupling

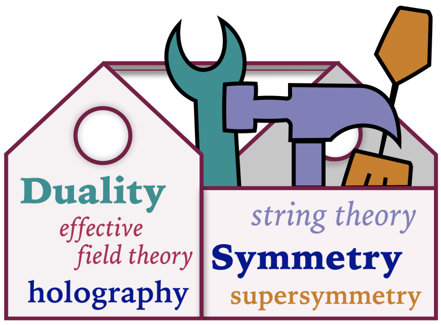
A Non-Technical Introduction

Dr. Emily Nardoni

Department of Physics and Astronomy, Vassar College, 124 Raymond Avenue, Poughkeepsie, NY 12604 USA

Abstract

This note serves as a gentle introduction to a modern research program in Quantum Field Theory, from one particle physicist's (biased) perspective. The goal is to explain some of the big questions that motivate a deeper study of the nature of Quantum Field Theory, and to provide context for a few of the tools that play a starring role in driving progress in the field. The content grew from a series of colloquia that the author gave in the subject. These notes are aimed at the level of an undergraduate physics major who has taken a first course in Modern Physics, and so has been exposed to core concepts in quantum mechanics and special relativity.



Contents

1 The Big Picture

1.1 A matter of scale

The job of a theoretical physicist is to construct theories that accurately describe nature. Crucially, any theory that one constructs will only be valid within a certain *regime of validity*. For example, classical Newtonian mechanics well describes much of the macroscopic phenomena of our everyday lives; given the positions and momenta of billiard balls or people or airplanes, their future dynamics can be determined from Newton's laws. However, Newton's laws are only an approximation of the more fundamental laws of quantum mechanics that apply to the microscopic world of the elementary particles. At scales where quantum effects dominate (roughly, when the de Broglie wavelength of the object is much smaller than the scale at which the measurement is taking place), the laws of physics are inherently *probabilistic*, as opposed to deterministic. In quantum mechanics, particles are described by wavefunctions, whose amplitudes give the probability of finding the particle in some particular state upon measurement.

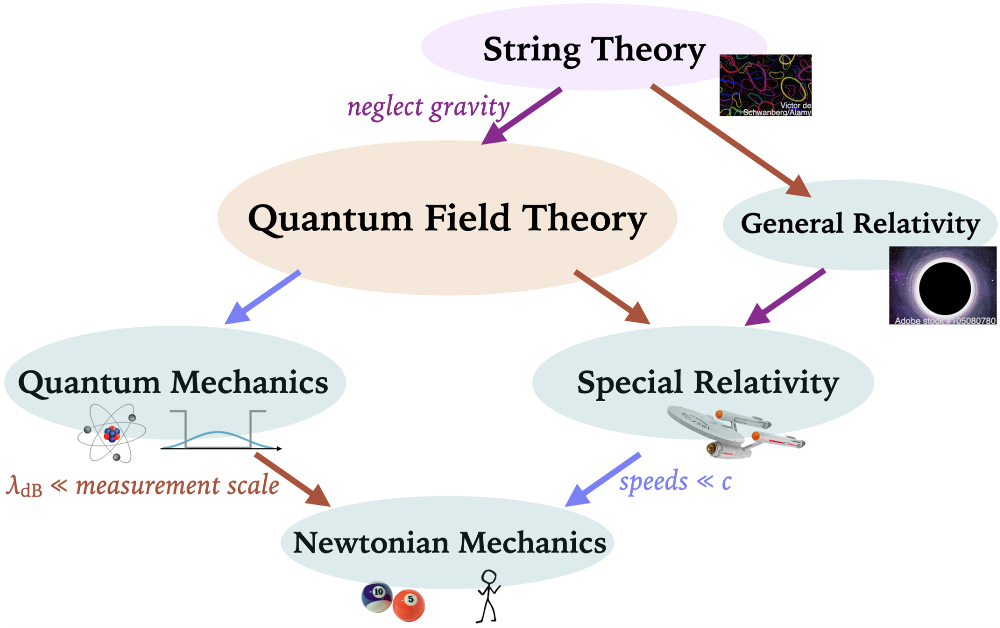


Figure 1: The theory tree.

On the other hand, when a billiard ball or rocket ship moves very fast, at speeds close to the speed of light $c \simeq 3 \times 10^8\, m/s$, Einstein's theory of special relativity is needed to describe its dynamics. With special relativity we uncover fundamental new features that appear far from classical—a highlight of which is the fact that mass and energy are two sides of the same coin, able to be converted back and forth into each other with the famous $E = mc^2$. It is only in the limit that objects are moving at speeds much smaller than the speed of light that the laws of special relativity reduce to the laws of classical mechanics.

The theory that marries special relativity and quantum mechanics is known as *Quantum Field Theory* (QFT). Quantum Field Theory provides a unified framework for accurately describing both quantum and special relativistic effects—it encompasses these other theories, reducing to them in the appropriate limits. Much as special relativity and quantum mechanics exhibit new features relative to classical mechanics, QFT also exhibits fundamentally new physics, some of which will be highlighted throughout this note.

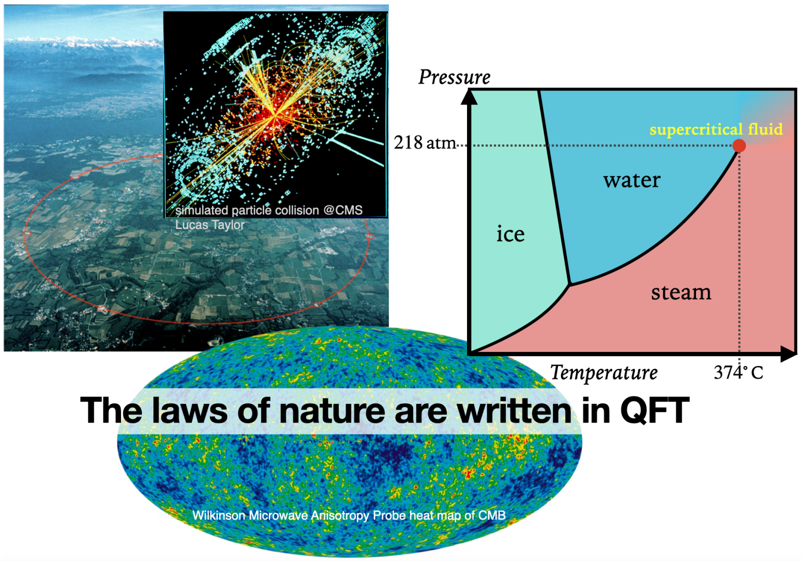


Figure 2: Some of the myriad of applications of QFT.

QFT is the natural language for describe a wide variety of systems in nature. It is essential in particle physics, for example in formulating the Standard Model of particle physics which describes the interactions of the elementary particles, and predicts to extremely high accuracy the effects of particle collisions at particle colliders like the Large Hadron Collider. In cosmology it is essential for describing the physics of the early universe and modeling inflation, with the aim in part to explain the observed Cosmic Microwave Background. It is used widely in condensed matter physics, to describe (for example) the fractional quantum hall effect, superconductivity, and the study of critical phenomena and phase transitions, such as the critical point on the phase diagram of water. Furthermore, it has many deep connections to mathematics, with advancements in theoretical physics driving new insights into mathematics, and vice versa. As QFT is at the heart of our best understanding of how nature works, a better understanding of its structure and application is an essential endeavor of modern research in theoretical physics.

It is important to mention from the start that QFT is not the end of the story in physics. The framework that consistently captures the physics of both QFT and Einstein's theory of *general* relativity, which describes how bodies move under the force of gravity, is *string theory*. String theory describes quantum theories of gravity; it is naturally formulated in 10 spacetime dimensions, and reproduces quantum field theories in the limit that gravitational effects decouple, and the extra dimensions are taken to be very small. The fact that this interconnection exists between string theory, theories of gravity, and QFT is extremely useful, and can be applied to very fruitful effect to glean insights into quantum systems. This idea of harnessing interconnections (or, *dualities*) between very different-seeming theories to learn new insights about the properties and dynamics of quantum fields is extremely powerful, and is a motif which appears over and over again in modern high energy theory research.

1.2 What is QFT?

The essential concept of Quantum Field Theory is that all of the elementary particles and forces in the universe are ripples of quantum fields that take values throughout all of space and time. Particles like the electrons, quarks, and photons are excitations of these fundamental fields.

Mathematically, a field is simply a quantity that has a value at each point in space (x, y, z), and time t. A familiar classical example of a field is the electromagnetic field, which consists of orthogonal vector-valued electric and magnetic field components, $\vec{E}(x, y, z, t)$ and $\vec{B}(x, y, z, t)$. Waves of electromagnetic fields propagate at the speed of light. Upon quantization, the excitations of the electromagnetic field have particle-like properties which we call *photons*. In Quantum Field Theory, *all* particles are excitations of more fundamental quantum fields—electrons are arise from excitations of electron fields valued at each point in space and time, and similarly for quarks, and so on.

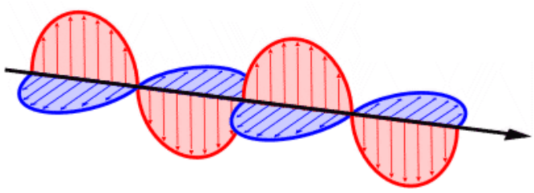


Figure 3: An electromagnetic wave.

In the same way that wavefunctions encapsulate probabilities in quantum mechanics, in field theory the field values tell you about the probability of the particle being found in a particular state. For example, if an electron field is localized around a point in spacetime, then the associated electron can be understood to be localized near that point. Fields can interact with each other locally, so that (for example) the act of two particles colliding is modeled by the interactions of their fields.

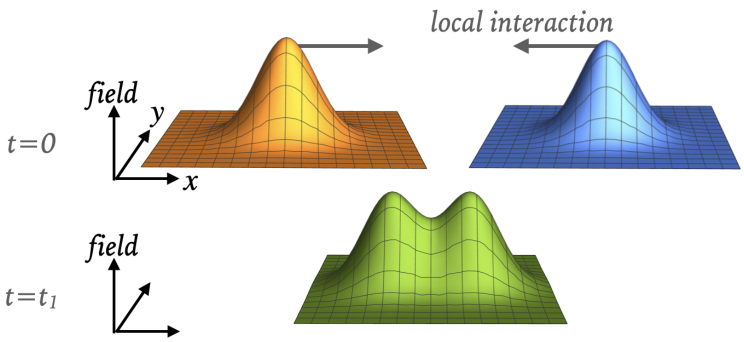


Figure 4: A schematic representation of two localized fields interacting.

The mattress model At the very heart of Quantum Field Theory is perhaps our favorite toy system in physics: the simple harmonic oscillator. QFT treats every point in all of space as a quantum harmonic oscillator, so that the field associated to a particle is composed of an infinite number of oscillators. In this quantum theory of electromagnetism (known as *quantum electrodynamics*, or QED)[1], one treats the normal modes of the electromagnetic fields as harmonic oscillators, which satisfy canonical quantization relations upon quantization.[2]

This fact that fields arise as collections of oscillators is behind much of the rich phenomena that QFT describes. For example, recall that the quantum harmonic oscillator has a zero point energy of $E = \frac{1}{2}\hbar\omega$; due to the uncertainty principle, the energy of a system described by a harmonic oscillator cannot have zero energy. It follows that even in completely empty space—the so-called *vacuum*—there is a nonzero energy density associated to this zero point energy. Since energy and mass can be converted into each other ($E = mc^2$!), this vacuum energy density can be converted into massive particle excitations. There is always a probability that particles

[1] The author apologizes in advance for the proliferation of acronyms. Theoretical physicists like acronyms.

[2] By "the quantum theory of electromagnetism", we mean that quantization methods can be used to turn the classical theory of electromagnetism described by Maxwell's equations into a quantum theory, which is required for computations involving electromagnetic interactions for which quantum effects are important.

can fluctuate in and out of existence at every point in space due to such quantum fluctuations. This is an amazing conclusion which leads to actual, measurable effects.[3]

Nature is interacting In the same way that the energy levels and wavefunctions of a particle in a quantum harmonic oscillator potential can be exactly solved in quantum mechanics, the free field is an exactly solvable system in Quantum Field Theory—the energy levels, analogues of the wavefunctions, and expectation values of physical observables can be exactly solved for analytically. But of course, in general nature is *interacting*. Including field interactions in this picture is like coupling the harmonic oscillators, so that the system becomes anharmonic. The system of anharmonic oscillators is no longer an exactly solvable system, and we need to use approximations and other methods to determine the physics.[4]

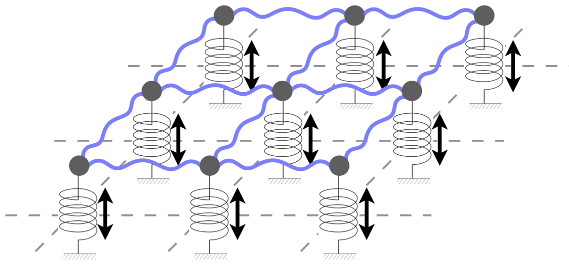


Figure 5: Coupled harmonic oscillators.

Naturally, how strong or weak the interactions are will determine what sorts of methods are best suited to describing their effects. The strength of field interactions are captured by parameters known as *couplings*. For example, for a quantum anharmonic oscillator with a potential of the form $V(x) = \frac{1}{2}kx^2 + gx^4$, the coefficient g to the quartic term is a coupling whose magnitude determines whether the anharmonicity is very strong or weak. Another (probably less intuitive) example of a coupling is the fundamental electric charge e in quantum electrodynamics. The fact that the value of the electric charge in nature is measured to be very small, $e \simeq 1.6 \times 10^{-19}\, C$, is a reflection of the fact that electrons and photons interact with each other relatively weakly. Life on Earth would be very different if e was a larger value![5]

When interactions between the fields are very weak and the couplings are small, then physicists have developed reliable tools for systematically capturing the effect of the interactions. These tools all fall under the umbrella of *perturbation theory*. The idea is that in a theory with a small parameter g, one can perform a Taylor series expansion in the small parameter, and thereby order by order in the expansion compute quantities of interest including successively larger powers of g. Such perturbative methods have been wildly successful at accurately predicting the physics of weakly-coupled QFTs such as quantum electrodynamics.[6]

For example, a quantity that one might wish to compute is the probability that two incoming electrons will scatter off of each other. This scattering process is possible precisely because photons and electrons interact with each other—photons are the particle-like excitations of electromagnetic fields, and act as the force-carriers of electromagnetic force, under which electrons are charged. Any particle that can interact with photons experiences electromagnetic force.

[3] The Casimir effect is the phenomenon that the energy of empty space produces a tiny force between two uncharged conducting plates—check out this Physics Today article for further reading.

[4] The fact that the free field is exactly solvable has a lot to do with the fact that the harmonic oscillator potential is quadratic, so that the resulting equations of motion for the wavefunctions contain at most two powers of x (in the quantum mechanics problem). Including higher order terms in the potential leads to a system which no longer admits exact solutions.

[5] Really, the dimensionless coupling which controls perturbation theory is $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}$. A famous quote by Richard Feynman expounds the smallness of this number as one of the great "mysteries of physics".

[6] A brief aside on language. *Quantum Field Theory* refers to the framework as a whole of describing systems by quantized fields. However, we also often refer to specific theories within this framework (with specific couplings, types of fields, *etc.*) themselves as quantum field theories, or QFTs. It is interesting both to better understand how to effectively use the framework as a whole, as well as how to apply it to learn about specific QFTs.

Since the interaction between the electrons and photons are weak, however, this probability can be accurately calculated in perturbation theory order by order in the coupling e. At leading order (assuming the fewest interactions between the incoming electrons and a photon), the amplitude is proportional to e^2, while at subleading order the amplitude is proportional to e^4, and so on. In physics, these scattering interaction probabilities are represented by *Feynman diagrams*; you can think of each of the diagrams in Figure 6 as a rule set forth by QFT that computes the probability of the interaction occuring. Since $e^2 \ll 1$, the second diagram contributes much less than the first diagram to the probability, and it is a good approximation to keep only the first terms in the expansion.

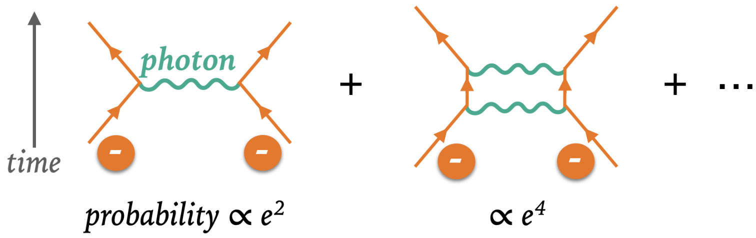


Figure 6: Electron scattering in perturbation theory.

However, these perturbative techniques fail for many interesting problems where couplings are not small and the fields are strongly interacting. (Clearly, if e^2 was order 1 then it does not make sense to series expand in powers of e^2.) In particular physics the quintessential example of such a strong coupling problem is that of the strong nuclear force; this is the force that involves the dynamics of quarks and gluons, explaining why the nucleus of the atom is bound together from protons and neutrons, and why protons and neutrons themselves are bound together from quarks and gluons. Formulating exactly how this happens is literally a million dollar question, put forward at the turn of the century by The Clay Institute.[7]

Overcoming this strong coupling problem is the foremost challenge of 21st century research in Quantum Field Theory. The big picture goal of the author's research program is to develop novel theoretical tools to solve for the dynamics of QFT at strong coupling; both for the purposes of developing toolkits that can be used widely in a variety of contexts, as well as in applying them to particular QFTs of interest in particle physics and string theory to learn about their physical properties. The rest of this note will describe a few of the various tools in the modern theoretical physicist's toolkit that are critical in addressing this challenge.

[7] This challenge can be more precisely formulated as proving the fact that the lowest energy state in a quantum theory of gluons has mass, and so is also known as "the mass gap" problem.

2 Getting Around the Strong Coupling Problem

2.1 Go with the (renormalization group) flow

It is an essential fact that physics depends on the scale at which you measure it. This was the idea with which this note began, and it underlies the modern way that physicists think about Quantum Field Theory. The mathematical framework behind this fact is captured by the *renormalization group (RG)*. The idea is that the couplings in a given QFT that parameterize the strength of interactions between the constituent fields are actually *not* constant (and so, commonly referring to them as coupling constants is a bit of a confusing misnomer!). The couplings change, or "flow" as a function of the energy scale at which they are measured, and the way in which they flow is baked into the formalism of the renormalization group.

To unpack this statement, the first concept to explain is that of an energy scale. If I wish to measure the height of a tall building, I can use a meter stick in order to obtain the height within an accuracy of $\sim \pm 1\,m$. If I wish instead to measure the position of a particle like an electron, clearly I need to resolve a much finer scale, and so I require a smaller fundamental unit of measurement. In general, when one measures an object's location by throwing light (or some other particle) at it and seeing what comes back, the distance that can be resolved is on the order of the wavelength of the light—the wavelength is like the meter stick, setting the scale of resolution. In order to measure the height or location of the building within reasonable resolution, we can do so with light of wavelength $1\,m$, which translates to an energy $E = 2\pi\hbar/\lambda$ of about 10^{-6} eV. On the other hand, an electron in a metal has a typical de Broglie wavelength on the order of nanometers, so that the energy of the light needed to resolve the electron is on the order of 10^3 eV. Therefore, measuring the location of the electron is a comparatively a high energy, or small-wavelength experiment, while measuring the height of the building is a low energy, or long-wavelength experiment.

Much like the relativistic world of the electron exhibits wildly different physics from the classical macroscopic world of cities and buildings, physics looks very different if I probe a system at high energies, which allows me to zoom in to very tiny distance scales, as opposed to low energies, zooming out to large distances. In analogy with the electromagnetic spectrum, physicists often say that probing a system at high energies corresponds to probing the ultra-violet, or UV, limit of the theory, while probing a system at low energies corresponds to the infrared, or IR, limit. We will use this language throughout this note. (Of course, the notion of high-energy versus low-energy is relative, and in any system of interest there may be many disparate energy scales which might be interesting to examine!)[8]

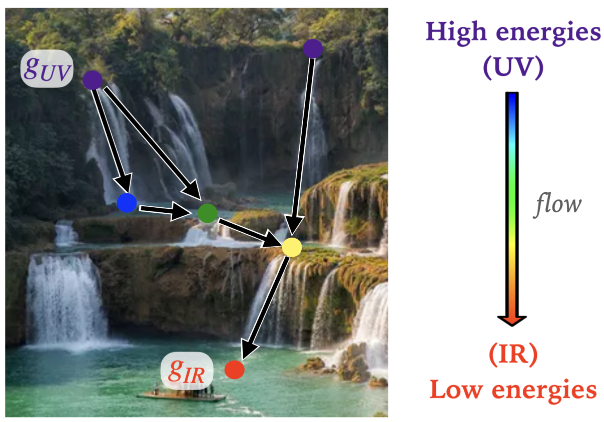


Figure 7: A waterfall analogy for the renormalization group flow of the couplings g in a QFT.

A mathematical description of the renormalization group is outside of the scope of this note, so we will instead use analogy to express the main ideas. A nice metaphor of RG evolution is water falling down a waterfall from high to low potential energy. The water might be

[8] As an aside, because particle physics experiments typically occur at very high energies, the field of particle physics is also known as *high energy physics*.

moving very quickly at the top, and depending on its initial velocities and how it's perturbed by the rocks it will fall down to different pools at lower potential energies, and continue on down the waterfall until it pools at the bottom in a lake (at lowest potential energy). Clearly, the pooled water at the bottom behaves very differently from the fast moving water at the top. Similarly, the value of the couplings in a QFT start from some initial value at high energies, but by the time we arrive at low energies far down the flow they might take some totally different value, so that the physics looks completely different.

In the example of quantum electrodynamics, we can think of the value of electric charge e that is measured in a typical experiment to be the low-energy value. However, if we were to probe the electron at much higher resolutions, or at very short distances, then the electron will actually have a slightly different (larger) value of the electric charge.

Emergent phenomena Since physics depends on the scale, the important question that needs to be addressed is, *what is the most useful way to extract information about the effective physics at a given scale that we care about?* Renormalization group flow is a kind of zooming out, and in this process there is a coarse graining of what the right effective particle content and interactions are that capture the short versus long distance physics.

A useful analogy is that of a forest.[9] If I probe the forest at short distance scales by walking through the forest, I experience the leaves, the dirt, and the individual trees. However if I zoom out and probe the forest from a large distance away, say by flying over the forest in an airplane, I can appreciate the larger structures—the forest as a whole, the shape of the river, the fact that it's placed in a valley and surrounded by mountains. It is true that the forest is still made up of many leaves on trees, but this description in terms of thousands of leaves is not *useful* to describe the features observed from the plane—one instead needs to talk about whole collections of trees. In the same way, collective, emergent behaviors can be totally different from microscopic behaviors. The goal of the theoretical physicist is to develop frameworks that allow us to extract the features of such emergent phenomena.

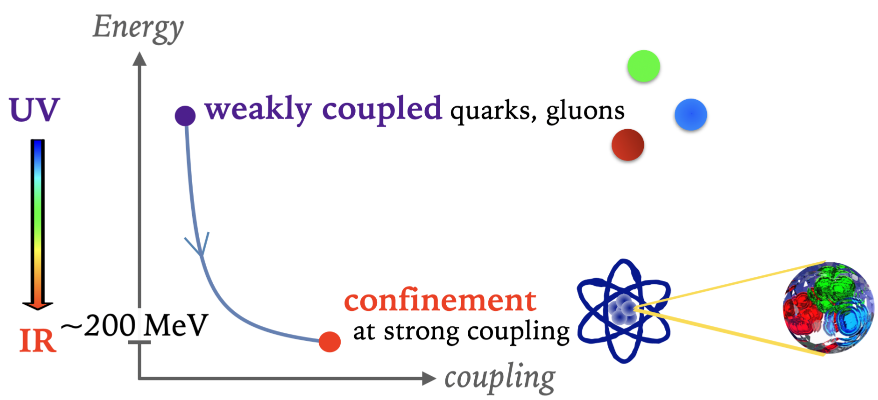


Figure 8: A schematic representation of the QCD phase diagram as a function of the coupling.

The strong nuclear force There are various options for how a QFT might behave under the renormalization group. In the example of quantum electrodynamics, the coupling is weak at low energies, but gets larger at high energies. The other quintessential example of a QFT in particle physics is the theory of the strong nuclear force between quarks and gluons, known as Quantum Chromodynamics (or for short, QCD). Figure 8 (schematically) depicts the QCD

[9] The author recalls first hearing a similar forest analogy made in a colloquium by Dr. John McGreevy.

coupling as a function of the energy scale at which the measurement is taking place, rotated so that the vertical axis goes from high to low energies. At high energies the coupling is small and the theory is weakly coupled. In this regime the quarks and gluons are weakly interacting, free to move about individually as they like. The physics of these free quarks and gluons can be probed at high energy particle colliders, and the results of experiments at these scales can be well described by perturbation theory.

However, as we dial down to lower energies with the renormalization group, at a scale of order ~ 200 MeV the coupling gets very large, and the theory becomes very strongly interacting. In this phase the quarks and gluons are bound up tightly into protons and other types of composite particles called hadrons, and the individual quarks cannot be separated out. This phenomenon is known as *confinement* of the color charge, since gluons are the force-carriers of the strong force (also known as the color force). Physicists have collected definitive experimental and numerical evidence for low-energy confinement in QCD, however it is an outstanding challenge as to whether we can gain a better *analytic* understanding of the emergence of these features, since they are the result of very large interactions. This is one of the holy grail strong coupling problems in particle physics.

Flow to a fixed point Another option for how a QFT can behave under renormalization group flow is it can flow to a point where the couplings no longer change with scale. Once the theory has arrived at such a "fixed point", one can keep dialing the energy scale down further and further and yet the value of the coupling will stay fixed, so that the physics of the theory in this regime is scale invariant. The type of QFT that describes such a scale invariant fixed point is called a *conformal field theory*.[10]

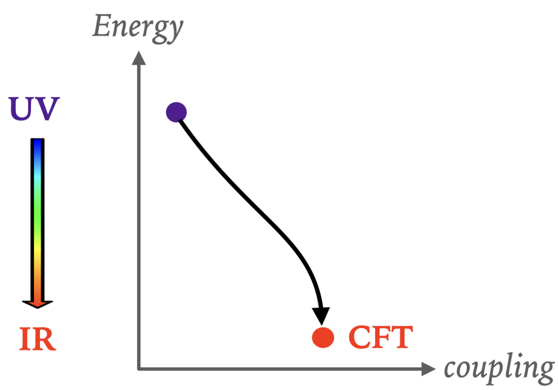


Figure 9: Flow to a conformal fixed point.

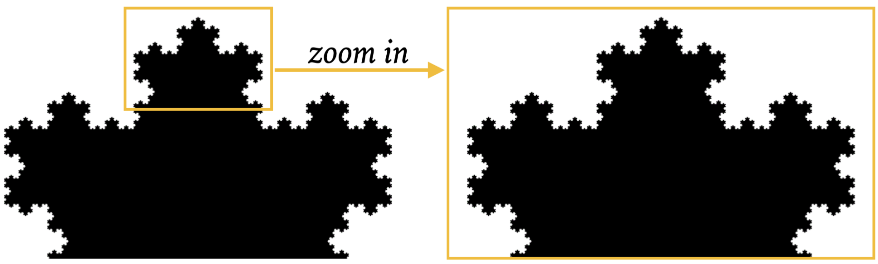


Figure 10: Like a CFT, a fractal exhibits scale invariance.

[10] It turns out that when a field theory is scale invariant, it is also invariant under a larger set of transformations known as conformal transformations, so that field theories with conformal invariance describe fixed points of the renormalization group. As we will touch on in the next section, conformal invariance is an important kind of symmetry that a field theory might have.

This scale invariant behavior is well visualized by a fractal, which has this property that as I continue to zoom in the fractal continues to look the same; in this way the behavior of a conformal field theory continues to look the same even as I zoom to smaller and smaller energy scales.

Conformal field theories (or CFTs) are extremely important throughout many subfields of theoretical physics, principally because they capture the physics of critical phenomena: CFTs model second order phase transitions. One example of such a transition is exhibited by the Ising model of a lattice of spins. Consider a 2d spatial lattice, where at every point on the lattice there is a variable that can take one of two values: up, drawn in blue in the figure, or down, depicted by white in the figure. Each spin is allowed to interact with its nearest-neighbor spins, in such a way that the energy of the system is lower when neighboring spins are aligned in the same direction.

The behavior of this system depends on the temperature, and one can think of decreasing the temperature from high to low values as a renormalization group flow from high to low energies. At high temperatures, the lattice spins are randomly aligned up and down, and the system is in the so-called disordered phase. At low temperatures, the spins all align in the same direction (since this minimizes the ground state energy). The Ising model is then a statistical model for ferromagnetic behavior, where the spin alignment represents the alignment of magnetic moments in a ferromagnetic material. In between the high-temperature disordered phase and low-temperature ferromagnetic phase, there is a critical temperature at which a phase transition occurs. When the system is tuned to this critical temperature, it becomes scale invariant, and the behavior of the system is described by a conformal field theory.

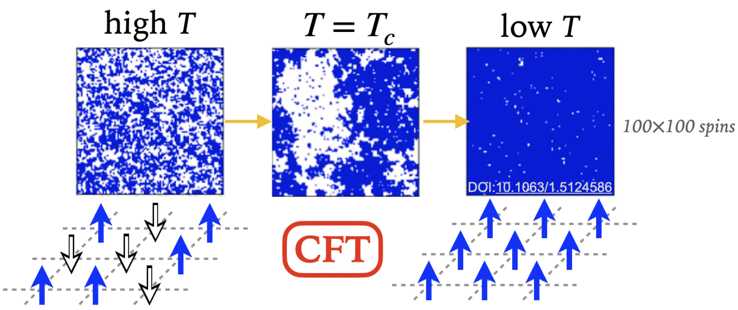


Figure 11: The Ising model on a 2d spatial lattice.

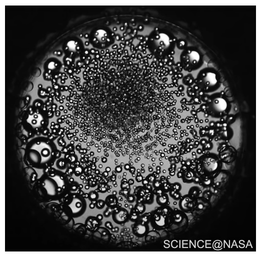


Figure 12: Supercritical fluid.

Another example of a second order phase transition is the critical point in the water phase diagram between water and steam, depicted in Figure 2. At high enough pressures, when water is heated to a certain temperature it becomes a supercritical fluid, which is a state of matter distinct from liquid or gas that exhibits scale invariant behavior. The physics of the supercritical fluid as water goes through this phase transition is modeled by a conformal field theory.

2.2 Duality and the interconnected space of QFT

Amazingly, it turns out that both of the seemingly very different critical points we just discussed—the supercritical fluid and the Ising model phase transition in two spatial dimensions—are described by exactly the *same* conformal field theory. This is a surprising fact underpinned by a deep phenomenon known as *duality*.

A duality is when two or more systems look extremely different at high energies—their microscopic fields and interactions and behaviors are completely distinct—but at low energies they describe exactly the same universal features, and are then described by exactly the same conformal fixed point.[11] In the waterfall analogy of renormalization group flow, this is when there are two or more QFTs at different starting points on the top of the waterfall, but which all flow to the same pool at the bottom. Thus, another reason that conformal field theories are very special amongst general QFTs is that they capture the universal behaviors of very different microscopic physics. This is a phenomenon that is also known in the condensed matter literature as *universality*.

These sorts of interconnections amongst quantum field theories are extremely powerful for addressing the strong coupling problem, because often dualities relate a strongly-coupled QFT with a weakly-coupled one. Suppose that QFT_1 in Figure 13 has (like QCD) very strong interactions at low energies, so that we do not have a good perturbative grasp on its properties, but QFT_2 is weakly interacting at low energies, *à la* quantum electrodynamics. Since QFT_1 and QFT_2 are dual, and thus described by exactly the same physics at low energies, we can use our traditional perturbative tools that are well-suited to the weakly-coupled dual QFT in order to learn about the IR limit of the strongly-coupled QFT_1. In this way, duality can provide a window into previously inaccessible strong coupling physics! One of the guiding principles of research in theoretical high energy physics is to harness such deep interconnections between QFTs—from renormalization group flows and dualities—in order to characterize the strong coupling properties of field theories.

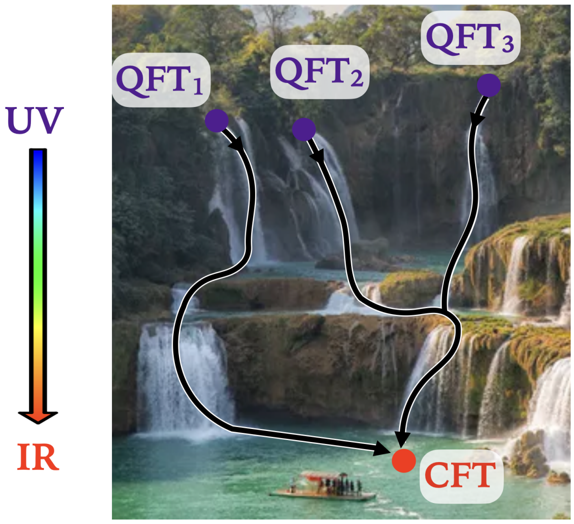


Figure 13: Dual quantum field theories.

Exploratory QFT We just saw how dualities between different quantum field theories that flow at low energies to the same conformal fixed point can be very powerful, since often it is easier to compute physical observables in one dual description than the other. The same logic applies to other kinds of dualities and interconnections. As we hinted at in the beginning of this note, the connections between string theory, theories of gravity, and quantum field theories also lead to interconnections that can be utilized to constrain the physics of strongly interacting QFTs. In the waterfall analogy, you can think of string theory and theories of quantum gravity as other points much higher up at the top of the waterfall, that can flow down and connect with a field theory at low energies. One of the big lessons of the last couple of decades of research in QFT and string theory is that these sorts of interconnections are completely *ubiquitous*.

It is useful to think about all the possible QFTs that have different field content, interactions, and properties, coming from all sorts of different constructions, as belonging to an abstract *theory space* of QFT. Different points in the blob of Figure 14 are each meant to represent a different theory with seemingly different properties. The interesting statement is that this theory space has deep, and often surprising interconnections. Dualities and renormalization group flows connect QFTs that might look on the surface wildly different from each other, and might individually be best suited for different types of toolkits, but nonetheless are secretly related in surprising ways which can be exploited. By exploring one corner of this space, we

[11] For the purposes of this note we will take this as the definition of duality, although other sorts of dualities also exist and are interesting!

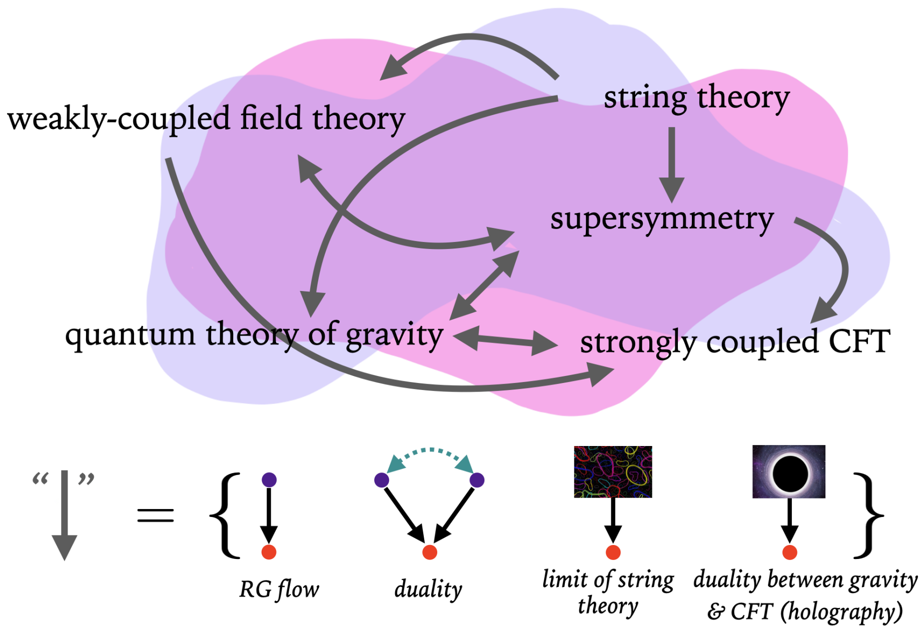


Figure 14: An abstract rendition of the interconnected space of QFT.

often actually learn important lessons about another corner. Experience has shown us that this sort of exploratory view of QFT is often extremely fruitful.[12]

[12] The author attributes the phrase *exploratory QFT* to their advisor Dr. Ken Intriligator, who has clearly influenced much of their point of view on physics.

3 The Power of Symmetry

As we will now review, symmetry has deep implications for the study of QFT.

3.1 What is symmetry?

Symmetries are everywhere in nature, from the reflection symmetry in a face, to the rotational symmetry of a snowflake. Symmetries can be either discrete, or continuous. For example, the pattern in Figure 15 has several discrete symmetries: it is symmetric under translation by 1 unit in the horizonal direction, as well as under translation by about 1.4 units in the vertical direction; furthermore it is symmetric under reflecting the image about its center axis. An example of a continuous symmetry is the rotational symmetry of a circle; the circle is invariant under rotations by not only discrete angles, but also infinitesimal angles.

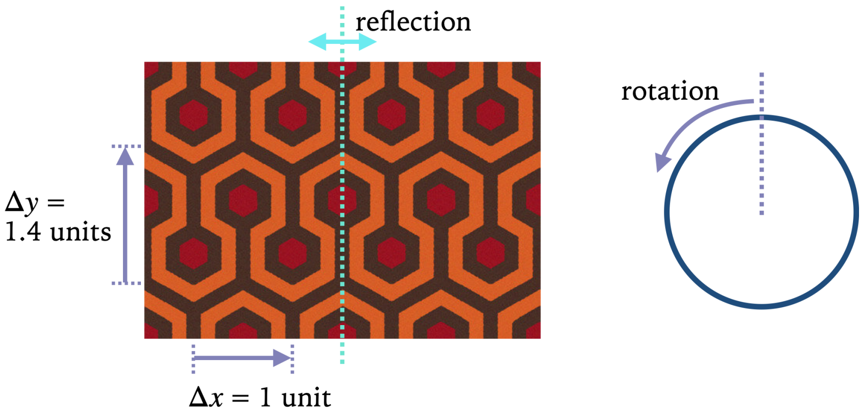


Figure 15: Examples of symmetries.

As humans we often use symmetry for its aesthetic appeal (*see:* Wes Anderson). On the other hand, symmetry in physics plays a fundamental role in formulating the physical laws of a system. In physics, by symmetry we mean a transformation that does not change the results of any possible experiment, leaving the underlying physical laws invariant. For example, Einstein's principle of relativity tells us that the laws of physics must take the same form regardless of where one is in spacetime, which translates into the statement that the laws of physics for a relativistic system are invariant under a set of spacetime symmetries (known as the Poincaré group). These include infinitesimal translations and rotations in space; whether a scientist performs an experiment standing in Poughkeepsie or in Tokyo, they must be able to use the *same* physical laws to predict the results of the experiment. A relativistic QFT is necessarily invariant under this set of spacetime symmetries.[13]

Another example of a symmetry that we have already mentioned is conformal symmetry. Conformal symmetry is a type of spacetime symmetry that includes scale invariance. As we stated earlier, quantum field theories that are invariant under conformal symmetry (*i.e.* conformal field theories) play a privileged role in theoretical physics.

[13] In this note we have motivated Quantum Field Theory as necessary to consistently describe the combination of quantum mechanics and special relativity, and so of course by this definition a QFT must obey the laws of relativity. However, the framework of QFT is actually still very useful in the limit where one or the other of these assumptions is relaxed; classical field theories that have not been quantized are extremely useful in many contexts, as are non-relativistic quantum field theories that relax the assumption that they describe relativistic physics. For instance, field theories can be applied to describe systems on a discrete spatial lattice, which preserves only discrete (rather than continuous) translations and rotations.

One more example of a spacetime symmetry that a theory might possess is supersymmetry. Supersymmetry acts on bosons to turn them into fermions, and vice versa. Then, a supersymmetric field theory has matter content that necessarily comes with equal numbers of fermions and bosons in equal mass pairs, so that a supersymmetry transformation leaves the theory invariant. The world as we have observed it is not supersymmetric (notably, the Standard Model does *not* have equal numbers of bosons and fermions), however it is possible that this is an effect of our "low scale" physics, and that the underlying laws of nature are in fact supersymmetric. Scientists are testing this hypothesis for signatures of supersymmetry at particle colliders like the Large Hadron Collider. Regardless of whether or not supersymmetry is actually realized in nature, however, there are good reasons as a theoretical physicist to study supersymmetric toy models of particle physics, some reasons of which we will touch on below.[14]

(*As a conceptual aside:* this discussion is likely a somewhat more abstract application of symmetry than you might be used to. In the example of relativistic Poincaré invariance, it is not that the physical world itself is invariant under translations or rotations—clearly Poughkeepsie and Tokyo are very different places—but rather, it is the equations of nature that are invariant under the symmetry. Of course, there are also situations where the physical state of the world described by a QFT is invariant under some symmetry even if the underlying equations are not (so that the symmetry is *emergent* from a system that did not originally possess the symmetry); or vice versa, where the underlying equations respect some symmetry but the physical state does not realize this symmetry (so that the symmetry is *broken*). We will soon see an example of this latter scenario, where the underlying laws of a system respect a symmetry, but a physical state of the system breaks it.[15] These possibilities are part of what make tracking symmetries a subtle task—in general one must determine (1) what are the underlying symmetries of the physical system of interest? and (2) how are these symmetries realized in the physical states of the system at different energy scales? Answering this second point especially can be quite intricate, but also reveal a lot of interesting physics!)

3.2 Lessons in utilizing symmetry

Let us return to the question of *why* symmetry is so powerful in theoretical physics for computing and constraining the properties of physical systems. Some reasons are as follows.

(1) First of all, **symmetries imply conserved quantities**, as formalized in Emmy Noether's theorem. For example, the invariance of a theory under spatial translations leads to the statement that momentum is a conserved quantity in this theory. Momentum conservation, energy conservation, angular momentum conservation, and any other conservation law that you encounter in your physics classes are each the result of an underlying symmetry (invariance under spatial translations, time translations, and rotations, for the aforementioned quantities). This is useful because conserved quantities are conserved *whether or not the interactions are weak or strong*, so can be used to constrain the strong coupling dynamics of a system.

[14] There are several reasons that supersymmetry in nature is an attractive possibility theoretically: for one, supersymmetry would unify all matter and forces, since it would relate the force-carrying bosons and fermionic matter particles by a symmetry; for another, most string theories are supersymmetric, and string theory is our current best candidate for a unified theory of physics at the highest energy scales. For the purposes of this note, however, supersymmetry is most valuable for another reason: for providing tractable toy models of particle/mathematical physics, where exact results can be obtained and new computational frameworks developed.

[15] For those that have taken a course in advanced classical mechanics, this is the difference between the *action* being invariant under a symmetry (thereby implying that the equations of motion are invariant), versus the *expectation value* of an operator in a particular state being invariant under the symmetry.

(2) Furthermore, symmetry is an indispensable principle for **distinguishing / characterizing phases of physical systems**.[16] For instance, in the earlier example of Ising spins on a lattice, the underlying physics is invariant under flipping all of the spins at once from up to down, which is kind of reflection symmetry. Doing this global reflection does not change any of the energies of states, equations of motion, or any other underlying conclusion about the system. However, at low temperatures the system chooses a lowest energy state where either all the spins are pointed up, or all the spins are pointed down, so that this choice breaks the underlying reflection symmetry. (In other words, in this phase the magnetization of the system can be pointed in the up direction or the down direction, but in either case the physics is ferromagnetic.) This is an example of *symmetry breaking*. This reflection symmetry breaking pattern is in fact the distinguishing feature of the low temperature ferromagnetic phase.

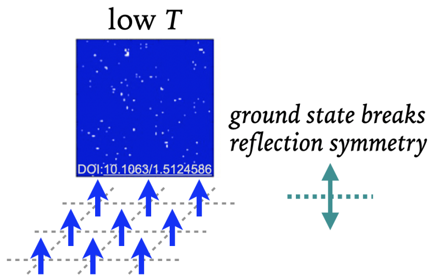


Figure 16: Symmetry breaking.

(3) **Symmetries constrain renormalization group flows**. This is because some symmetry data associated to a QFT is actually *independent* of the energy scale, and so is completely invariant under renormalization group flows. Basically, associated to a given QFT with some set of symmetries is a collection of numbers, which for concreteness we can refer to as $\mathcal{A}$.[17] Suppose that the QFT is weakly coupled at high energies, or in some particular duality frame. Then, it is straightforward to compute these numbers $\mathcal{A}$ at that scale or in that frame, using standard perturbative techniques. But then comes the power of this method: since these numbers do not change under renormalization group flow, *even if the QFT is very strongly coupled* at low energies or in a dual description, its symmetry data $\mathcal{A}$ has already been determined and applies to all such limits. Very few quantities in physics are completely scale independent, and so this data gives a rare glimpse into the properties of the strongly coupled phase of the theory.

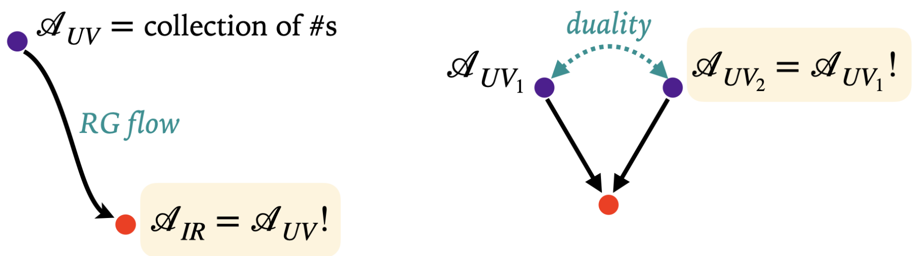


Figure 17: Using symmetry data to constrain RG flows.

(4) **Enough symmetry can allow for exact solutions**. There is a precise sense in which the more symmetry a system has, the easier it is to exactly solve for its properties without needing to resort to approximate methods. A good metaphor for the utility of simplified models for complex phenomena is the spherical cow. Cows are complicated shapes; they have four legs, two ears, small hairs all over, *etc.* In order to accurately describe the motion of a cow, all of these features should be taken into account—how they affect the wind resistance against the cow, its rhythm of motion, and so on. It would be difficult to make any sort of useful prediction taking into account all of these complex features. So instead, one might first make a simplifying approximation and consider the cow to have spherical symmetry. We can

[16] This is known as the *Landau paradigm* of the classification of phases of matter.

[17] These numbers are called *'t Hooft anomalies*, and their computation and characterization play a starring role in modern high energy theory research. For reference, as of writing, 13 of the author's 16 published articles concern in large part 't Hooft anomalies.

easily exactly calculate the motion of a sphere on flat ground when subjected to various forces. Then, once we've understood the motion of the spherical cow, we can start breaking the spherical symmetry—first in small ways, and then in large ways—to understand how to best adapt the computations.

Figure 18: Spherical cows.

An example of using additional symmetry to concretely compute a strong coupling quantity is as follows. Conformal field theories have associated to them numbers known as *central charges*, which provide a count on how many bits of information are needed to effectively characterize the system at a given energy scale. (Physicists call this information needed to specify a system *degrees of freedom*.) Recall from our earlier discussion that renormalization group flow can be viewed as a kind of coarse graining. In this coarse graining from high to low energies we generally need less information to specify the system; when walking through the forest one needs to describe the positions of thousands of leaves and hundreds of trees, while from far distances away we might only need to specify that there is 1 forest, with 2 mountains and 1 stream. By analogy, if a CFT at high energies and a CFT at low energies are connected by a renormalization group flow, then I would expect the CFT at low energies to have a relatively *smaller* central charge, since it should have less effective variables to describe. This expectation is in fact born out; there are theorems that the central charges of conformal field theories must decrease under renormalization group flows. The central charges is then an extremely important window into the properties of a strongly-coupled CFT, since we might not have any other means to get an idea of how many degrees of freedom it describes.

Unfortunately, it is not known in general how to compute the central charge of a generic strongly interacting conformal field theory. However! If the CFT has the larger symmetry of supersymmetry, then supersymmetry provides enough constraints that the central charge can often be computed *exactly*—there is a straightforward calculus-based extremization algorithm for doing so. Applying this algorithm and computing the central charge of a strongly interacting CFT is a useful tool for gaining a window into its physics.

Supersymmetry often allows for exact, analytic solutions of the properties of quantum field theories, another important example of which is as follows. In a supersymmetric version of a theory that describes the dynamics of gluons, it is possible to derive that the theory confines at low energies, and follows a particular symmetry breaking pattern that is reminiscent of what happens with the real world strong nuclear force. Thus, the dream of analytically deriving confinement and other strong coupling properties of a QCD-like theory is realized in a supersymmetric version of the theory! Examining whether or not the lessons learned using supersymmetric QFT can be extended to glean new insights into non-supersymmetric theories like quantum chromodynamics is an interesting question under active research.

Shining a light on the space of QFT We have emphasized throughout this section that the more symmetry a field theory possesses, the more handles exist for computing its distinguishing observables. QFTs with large symmetries can be very constrained, and some of their features can be determined exactly even in very strongly interacting regimes by using the interconnections between QFTs and symmetry-based methods.

Extrapolating from these observations, a very useful perspective is to view these more tractable QFTs with large symmetries—such at conformal field theories and supersymmetric field theories—as privileged lampposts amongst general QFTs, which we can employ as a testing ground for probing physics at strong coupling, thereby shining a light on some corners of the space of more general QFT. The idea is that we first develop new computational frameworks in these highly-symmetric corners where there are more tools at our disposal. Then, we slightly break symmetries, perturbing away in a controlled fashion from these special points in order to learn how to adapt our toolkits to the study of more realistic QFTs, eventually carving out our understanding of more and more of this space. This perspective underlies much of the research in modern particle theory and mathematical physics.

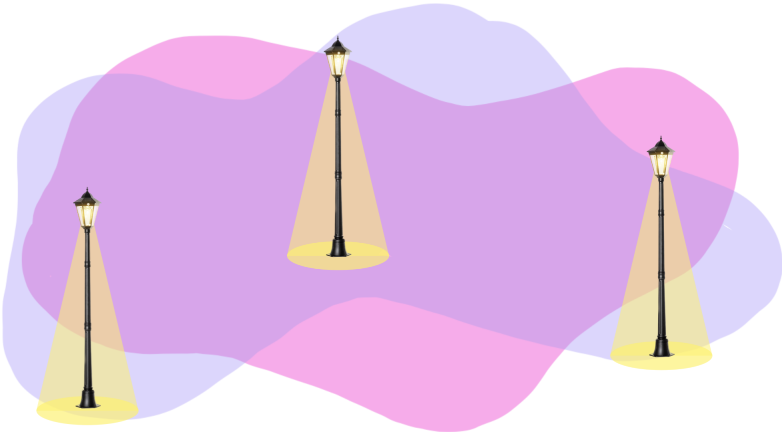


Figure 19: Theories with large symmetries are useful lampposts in the space of QFT.

4 What String Theory Can Do For You

This note began with the statement that QFT is the fundamental framework that unifies quantum mechanics and special relativity. String theory is thought to be the more fundamental theory that unifies QFT and general relativity, so that it is the consistent framework for describing quantized theories of gravity. Before concluding, we will briefly comment on how string theory can be employed to teach us general lessons about quantum field theories.

From particles to fields to strings String theory describes all the particles and forces in the universe in terms of modes of tiny vibrating strings, so that each particle is identified as a vibrational mode of an elementary string. This string-like nature of particles would only be evident at very large energy scales / tiny distances, on order of the Planck scale $\sim 10^{-35}\, m$.

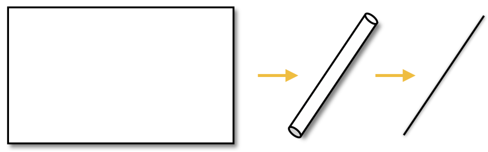


Figure 20: Compactifying a dimension.

String theories require extra spacetime dimensions. Our macroscopic world consists of three spatial dimensions, so that we live in four-dimensional spacetime. String theory is defined in 10 spacetime dimensions (9 space and 1 time), so that if string theory is actually realized in nature, six extra dimensions must be curled up very small so that we do not notice them. A good way to conceptualize the idea of extra dimensions is to think about a piece of paper. The paper has 2-spatial dimensions, but if I roll it up into a cylinder with a very small radius, it will look effectively like a 1-dimensional line. If the paper is rolled up tightly enough, an ant constrained to live on the line might never guess that there is a curled up extra dimension beneath its feet!

Quantum field theories arise from limits of string theory in which the gravitational degrees of freedom are decoupled. The QFTs of most interest to our real world—like quantum chromodynamics, quantum electrodynamics, or condensed matter systems like the Ising model—are (of course) defined in 4 or lower spacetime dimensions, and so arise from limits of string theory that involve compactification (rolling up the paper). These limits describe a sort of *cross-dimensional renormalization group flow*, where at high energies the system is described by a string theory (something way up high on the waterfall), and at low energies after taking the compactification/decoupling limit the system is described by a quantum field theory.

Turning the crank Like the other interconnections we have discussed in this note, this interconnection between string theory and field theory has proven very useful for learning about general properties of QFTs. One highly utilized point of view is to use string theory as a tool for generating different interesting QFTs, by taking these different limits. From this perspective, string theory can be viewed as a sort of black box. The input to the black box is a particular string theory with some configuration of dynamical objects (called branes), and a space to compactify on. Turn the crank, and as output the black box generates a (generally supersymmetric) quantum field theory, furnishing another point in the ever-expanding theory space of Figure 14.

One reason this perspective is interesting to pursue is that a generic QFT output by this procedure is very strongly interacting, and so furnishes a potential testing ground for developing techniques suited for strongly coupled physics. In fact, this procedure typically outputs QFTs that have *no* tunable coupling whatsoever, so that there is no traditional sense in which there

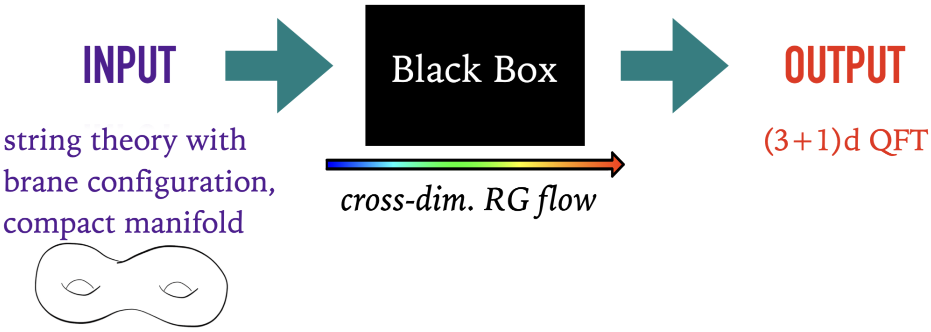


Figure 21: Generating QFTs from string theory.

exists a free field limit in any region of their parameter space. The fact that such fantastical QFTs exist is an amazing and surprising conclusion of the last 30 years of research in string theory—clearly, the space of QFT is much wilder than we might have thought!

Another reason this perspective is so powerful is that it allows a *geometrization* of QFT data, since the effective degrees of freedom and symmetries of the field theory are organized geometrically in the higher-dimensional string theory. This allows for the opportunity to develop geometric and topological tools to understand and compute field theory observables. One of the goals in this field is to understand precisely how the geometry encodes the symmetries and related symmetry data of the QFT, and how to systematically extract this data using techniques applicable even at strong coupling. Theoretical physicists have made a lot of progress in this direction, but there is still much to understand.

5 Outlook

Let us summarize. In this note we have reviewed why Quantum Field Theory is an essential framework in modern physics, and that it is necessary to develop novel, creative tools to solve for the dynamics of QFTs in strongly interacting regimes. The modern perspective of QFT is based on the fact that the useful effective description of the physics changes with scale (along renormalization group flows), so that collective phenomena are emergent at low energies. Renormalization group flows lead to deep and surprising interconnections (like dualities!) between QFTs, which theoretical physicists harness to characterize their strong coupling properties.

Furthermore, symmetry is an indispensable tool for both organizing the space of possible QFTs, as well as computing and constraining their distinguishing observables. We have seen that it is often fruitful to first analyze systems with *large* symmetries (especially supersymmetry, or conformal symmetry), where we have more tools at our disposal, and then to study how to extend our toolkits to theories with less symmetries. Understanding how to fully characterize the symmetry structure of a quantum field theory is thus one of the most interesting questions driving research in the field. There are many exciting related questions that theoretical physicists hope to answer, a few of which are as follows.

- What is the most general classification of phases of quantum systems? (Or in other words, how do we utilize symmetry in the most general possible way to expand the Landau paradigm?)
- What are the most restrictive constraints symmetries place on the strong coupling dynamics of QFTs?
- Can we gain a deeper analytic understanding of confinement in quantum chromodynamics?
- Can we systematically classify the appearance of emergent symmetries in renormalization group flows?
- What is the space of all possible conformal field theories—how do we characterize it and differentiate between different universality classes, and what is the most useful grading on this space? (This question is more tractable with supersymmetry, so a refined version is: can we construct all the possible consistent superconformal field theories?)
- Can every QFT be obtained from some limit of string theory?
- Is there a systematic way to generate all possible dualities between QFTs (perhaps within the context of string theory?)
- How is the full symmetry structure of a QFT that is obtained from string theory encoded in the geometry?

If the history of particle theory research has taught us anything, it is that in answering these questions we are certain to learn more fundamental truths about the nature of the world around us.