Symmetry in Physics Lecture Notes

Physics 375, Spring 2025

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Contents

1 Introduction

Possibly helpful resources:

- **MOST RECOMMENDED:** *A First Course on Symmetry, Special Relativity, and Quantum Mechanics* by Kunstatter and Das, Chapter 2 through 2.4.1. (Chapter 1 is also nice background reading material.)
- The article *The role of symmetry in fundamental physics* by physicist David Gross (posted on the webpage) is a quite nice, colloquium-level overview of the topic from his perspective.
- The Feynman Lectures on Physics contain a very nice overview of symmetry in physical laws, at this link.
- For some supplementary reading, the Stanford Encyclopedia of Philosophy has a nice article about Symmetry and Symmetry Breaking at this link, paying special attention to the connection between physics and philosophy. It's an interesting read, that you might wish to return to at the end of this course.
- An online gallery of M.C. Escher works gathered on the theme of symmetry, at this link.

1.1 What is symmetry?

Human fascination with symmetry dates back at least to the ancient Greeks and others who were interested in the symmetry of objects. Indeed, the word symmetry is borrowed from the ancient Greeks, for whom its importance was evident in the design of their buildings and appreciation of beauty. Symmetry plays an important role in art and aesthetics: shapes, paintings, sculptures, buildings, *etc.* tend to be more pleasing to the eye if they have some degree of symmetry. In art, symmetry appears in the concept of balance.

An example that is prominent in nature, art, and physics is bilateral, or mirror-image symmetry, *i.e.* the mirror-image reflection of an object about some axis. For example, human bodies and faces look almost the same upon reflection about the vertical axis.

Figure 1: Examples of bilateral symmetry in art (*Swan, Rush and Iris* by Walter Crane) and nature.

To start, it is helpful to have a general definition of what we mean by a symmetry. (We will get more formal later!)

Definition 1.1. A **symmetry operation** is the transformation of an object that leaves the object unchanged, or invariant.

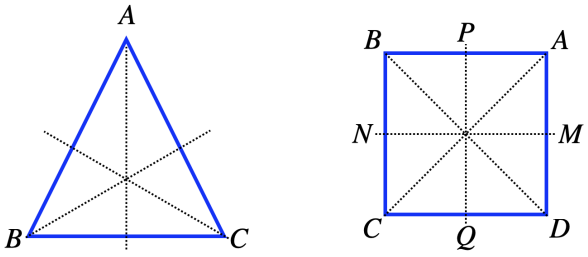


Figure 2

To get a better sense of this definition, it is useful to think about symmetries of geometrical objects. For example, consider an equilateral triangle. One type of symmetry operation is rotation. Rotation by 120°, one third of a full 360° rotation, leaves the triangle invariant. So does rotation by 240°. Of course, rotation by 360° is equivalent to doing nothing at all. Another type of symmetry operation is reflection. The possible reflections of an equilateral triangle are the reflections about each of the three axes going through the three vertices.

Another example are the symmetries of a square. Again, there are two types of symmetry operations: rotations and reflections. In this case, a symmetry operation is reflection by units of 90°. Rotating by 90°, 180°, and 270° each yield unique symmetry transformations. Meanwhile, reflection across the following axes yield further unique symmetry transformations: the vertical axis bisecting the square in two halves vertically; the horizontal axis bisecting the square in two halves horizontally; the diagonal top-left-to-bottom-right axis; and the the diagonal top-right-to-bottom-left axis. The square has four axes of symmetry, as there are four different ways to fold it and have the edges match onto each other.

Already in these examples, we can see one property that classifies different types of symmetries: symmetries can be **discrete** (like, reflection symmetry, which involves a binary operation), or **continuous** (like, rotational symmetry, which is valid for infinitesimal rotations).

1.2 The role of symmetry in physics

In physics, there was a paradigm shift in the early 20th century where symmetry principles were understood to be foundational to the way we formulate fundamental physics. One can argue that this started with Einstein, who understood in his formulation of special relativity that symmetry principles should be considered as fundamental, themselves constraining the dynamical laws of nature, rather than some output of the dynamical laws. Einstein recognized that relativistic invariance dictates the form of Maxwell's equations, and proceeded to elevate the symmetry of Maxwell's equations to a symmetry of spacetime itself, leading to what one might call the *geometrization of symmetry*.

The fundamental laws of physics are given by mathematical equations. In classical mechanics, these include Newton's second law $F = ma$, Newton's law of gravitation (that the force between two masses is inversely proportional to the square of the distance between them, $F = Gm_1m_2/r$), *etc.* In Einstein's theory of special relativity, these include the Lorentz trans-

formation law for transforming coordinates of events between two inertial frames of reference. In Maxwell's theory of electromagnetism, these include Maxwell's equations, for instance $\nabla \cdot B = 0$.

In physics, we usually discuss symmetries of a *physical theory*, by which we mean symmetries that leave the physical laws (*i.e.*, equations of motion governing the system) invariant. Going back to our general definition of a symmetry operation, the "object" is typically the system of equations describing the physical laws, so that the symmetry acts on these laws to give back precisely the same set of laws.

For example: Maxwell's equations are invariant under spatial translations,

$$x \to x + \Delta x$$

This type of symmetry is a little different than the geometric type of symmetry of objects we were just discussing. Translations are a symmetries of spacetime, or **spacetime symmetries** — the symmetry operation acts on the spatial coordinates x, so that the laws of physics that depend on those spatial coordinates remain unchanged. Rather, in the geometric examples the symmetry was acting on a physical shape / geometric object. Both types of symmetries are of interest in physics.

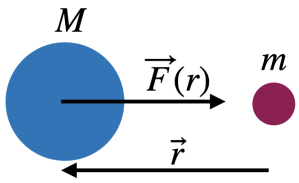
So, specifically why are symmetries so useful in physics? There are many reasons, four of which we will discuss below.

(1) Symmetry as a guiding principle Symmetry is often used as a guideline for constructing new theories. For instance, symmetry played a pivotal role in the formulation by Einstein of both special relativity as well as general relativity, in that he elevated spacetime symmetries as fundamental to the formulation of the theories. The Standard Model of particle physics describing the fundamental particles is organized around symmetries, and has been successful at predicting new particles based on these symmetries.

In general, certain symmetries are expected to be respected by *any* viable theory of nature; for instance, the fact that the laws of physics should be the same anywhere and at all times, and should not changed if I step to the left, or rotate my laboratory by 10°, should be true of any fundamental theory.

As an aside: what do we mean when we say that "a theory" should obey some symmetry? The job of a theoretical physicist is to construct theories that accurately describe nature. Any theory that one constructs will only be valid within a certain *regime of validity*. For example, classical Newtonian mechanics well describes the macroscopic phenomena of our everyday lives, but Newton's laws are only an approximation of the more fundamental laws of quantum mechanics that apply to the microscopic world of the elementary particles. On the other hand, when objects are moving close to the speed of light c, Einstein's theory of special relativity is needed to describe their dynamics. In the appropriate limits (for example, at speeds $\ll c$), these more "fundamental" theories should reduce to Newtonian mechanics. Some symmetries we might expect to be true of *all* theories of nature, while some we might expect are only valid in certain limits. When there is a good reason to believe that a symmetry will be or should be obeyed in some limit in which your theory is valid, that symmetry provides constraints on the theory.

(2) Symmetry and conserved quantities One of the most important features of symmetry is its connection to conserved quantities.



Why are symmetries related to conserved quantities? Let's consider a nice conceptual example you have probably seen before: a particle of mass m in a spherically symmetric gravitational potential, under the influence of a massive object M. Newton's law of gravity leads to the following force that the mass M applies on the mass m:

$$\vec{F} = -\frac{GMm}{r^2}\hat{\vec{r}} = -\frac{GMm}{r^2}\frac{\vec{x}}{|\vec{x}|}$$

where $\vec{r}$ is the displacement vector between the two bodies (pointing from the mass m to the mass M), and $\hat{\vec{r}}$ is the unit vector between the bodies. The gravitational force is said to be spherically symmetric because it doesn't matter which angle we are looking at: it only feels the dependence on the radial distance r between the objects and nothing else. In particular, we can rotate the mass m in θ and ϕ and the dynamical problem remains exactly the same.

This spherical symmetry buys us a conservation law: conservation of angular momentum. Consider the angular momentum of the mass m,

$$\vec{L} = \vec{x} \times m\vec{v}\,.$$

Taking the time derivative,

$$\frac{d\vec{L}}{dt} = \vec{v} \times m\vec{v} + \vec{x} \times m\vec{a} = 0 + \vec{x} \times \vec{F}\,.$$

The first term in the cross product is zero since a vector cross itself is zero (they are parallel). But for the case of the spherically symmetric potential, the second term is *also* zero, since the gravitational force $\vec{F}$ is parallel to $\vec{x}$; the force points in the same direction as the position vector. Therefore,

$$\frac{d\vec{L}}{dt} = 0\,.$$

We can immediately integrate this equation to obtain that $\vec{L}$ is some constant vector. In other words, angular momentum is a constant of motion — a conserved quantity — in the presence of a spherically symmetric potential. The angular momentum of any particle in the presence of such a potential cannot change without the action of some non-spherically symmetric external force.

Exercise 1.1

Suppose a single particle of mass m is in a 2d potential $V(r)$ that is independent of the angle θ. Use the fact that $\frac{\partial V(r)}{\partial \theta} = 0$ to show that the z-component of angular momentum, $L_z = (\vec{r} \times \vec{p})_z$, is time-independent. (*Note:* you will need to use the equations of motion.)

The general statement is that *symmetries lead to conserved quantities.* Every conservation law that you have every come across — conservation of momentum, energy, electric charge, *etc.* — is the consequence of a symmetry underlying the dynamical equations of motion. This

is formalized in Emmy Noether's theorem, which we will discuss later in this course. More generally, the quantum mechanical notion of a particle and its charge — from the electric charge of an electron to the color charge of quarks — arises from the fact that the underlying equations that describe them can be cast in the language of symmetries. Nature organizes itself using the language of symmetries.

(3) Using symmetry to simplify problems Practically speaking, symmetry is an extremely useful tool for simplifying calculations. If a system has some symmetry, then the equations can typically be greatly simplified by choosing variables that are best adapted to these symmetries. For example, perhaps you remember studying the radial wave equation for an electron in the hydrogen atom in Quantum Mechanics. The radial wave equation, written in spherical coordinates, is significantly simpler than studying the 3d Schrödinger equation in Cartesian coordinates, precisely because the Coulomb potential has a spherical symmetry.

A simple functional example is using the symmetry (or anti-symmetry) of an integrand to simplify the evaluation of an integral. The integral of an anti-symmetric function over a symmetric domain of integration is always zero; for instance,

$$\begin{aligned}\int_{-b}^{b} \sin x\, dx &= \int_{-b}^{0} \sin x\, dx + \int_{0}^{b} \sin x\, dx = -\int_{0}^{-b} \sin x\, dx + \int_{0}^{b} \sin x\, dx \\ &= -\int_{0}^{+b} \sin(-x)\,(-dx) + \int_{0}^{b} \sin x\, dx = -\int_{0}^{b} \sin x\, dx + \int_{0}^{b} \sin x\, dx = 0\end{aligned}$$

where in the second line we changed variables $x \to -x$, and used $\sin(-x) = -\sin x$. On the other hand, the integral of a symmetric function over a symmetric domain of integration can be simplified to twice the integration over half of its domain; *e.g.* $\int_{-b}^{b} \cos x\, dx = 2\int_{0}^{b} \cos x\, dx$.

Another useful application of this idea is to consider systems that are close to symmetric. If one assumes that a system has more symmetry than it actually has, so that is has an **approximate symmetry**, then the equations simplify, and the problem might become solvable. If it's a good approximation, then we can get a good sense for how the messier, real system would behave.[1] Often, methods can then be developed that perturb around the symmetric solution in order to capture the leading order corrections. In general, it is interesting to consider symmetries of nature that are both exact and approximate.

(4) Symmetry and classification Furthermore, symmetry is an indispensable principle for distinguishing or characterizing the phases of physical systems. This arises from characterizing when underlying symmetries are either preserved or broken by the state of a system. This phenomenon is called **spontaneous symmetry breaking**.

The idea is as follows. The laws of physics are thought to be the same everywhere in space and at all times. However, we know just by looking around us that the universe decidedly does

[1] This notion is captured by the hyperbolic idea of the *spherical cow*.

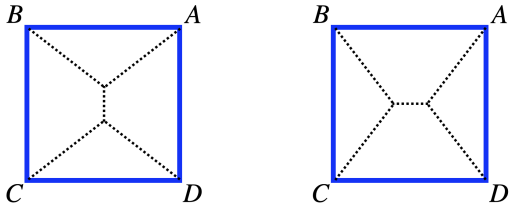
not look the same everywhere or in every direction. Given the high degree of symmetry obeyed by the equations describing the universe, why is it that the observed structure of the universe lacks symmetry?

The answer is that just because the equations of motion obey some symmetry, the physical system itself (*i.e.* the solutions to the equations of motion) does not need to obey that symmetry. This is exemplified by the following exercise, taken from Kunstatter-Das.

Exercise 1.2

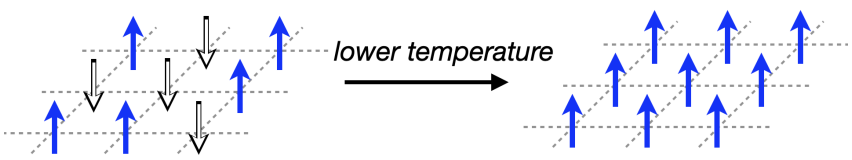
4 towns (A, B, C, D) are located on the 4 corners of a square with sides of length $a\,km$. An engineer is called in to design a road that connects all four towns at minimal cost.

The simplest possibility is to join the towns directly along the perimeter of the square, so that the total length of the road is $4a$. This respects the underlying symmetry of the problem, which is invariant under 90° rotations, but results in a road of length $4a$. It turns out that there are 2 optimal options (pictured below) that respect rotations by 180°, but break rotations by 90°. Calculate the length of the road for each of these options.



In the exercise, the two optimal options have the same shortest length of road, but break the 180° rotational symmetry of the underlying system.

A physics example of this phenomenon is the example of Ising spins on a lattice, which is a statistical model for ferromagnetic behavior. The underlying spin system has a reflection symmetry upon flipping all spins at the same time — such an operation does not change the energy or equations of motion of the system. At low temperatures, the ground state energy of the system is minimized by the configuration in which all spins are aligned in the same direction, either up or down. Treating the spins as little magnetic moments, this means that the ground state has a net magnetization which is either pointed up (if all the spins are aligned upwards) or down (if all the spins are aligned downwards). The fact that at low temperatures the underlying reflection symmetry is broken by the ground state (since nature chooses either the spins to be pointed up or down) is an example of spontaneous symmetry breaking. In fact, this symmetry breaking pattern is *the* distinguishing feature of the low temperature ferromagnetic phase. This classification of phases of matter by their symmetry breaking patterns is known as the *Landau paradigm of phase transitions*, and plays a significant role in the study of condensed matter and high energy physics.



1.3 The mathematical structure of symmetries

The mathematical structure that underlies the study of symmetries is known as **Group Theory**, and in the beginning of this course we will spend some time studying it. Formalizing / quantifying the nature of symmetry using the language of group theory will allow us to apply our study in many different examples in physics. With this formalization, we will be able to make precise many of the imprecise statements we've made throughout this introduction.

The connection between symmetry and mathematics is so important that often, a "Symmetry in Physics" course is taught as basically a "Group Theory for Physicists" course. However, this is decidedly **not** the goal of this course. While we will need to cover a fair bit of group theory in order to discuss symmetries in various contexts, this is truly meant to be a physics course that involves some math, not a math course that involves some physics, geared towards physics undergraduates. We will always stay grounded in physical examples, and not cover the mathematics with anywhere near the rigor that a traditional "Symmetry in Physics" course (or especially a true math course) might cover. For this reason, this course is forging somewhat new ground — I do not know of an undergraduate course that covers this material in this way. (If you do... please tell me!)

——— *End Lecture 1.*

2 The Language of Symmetry: Group Theory

Possibly helpful resources:

- **[MOST RECOMMENDED:]** *A First Course on Symmetry, Special Relativity, and Quantum Mechanics* by Kunstatter and Das, Chapter 3 sections 3.1-3.4 for the basics on group theory.
- *A First Course on Symmetry, Special Relativity, and Quantum Mechanics* also contains a nice overview of vector calculus, linear transformations, and matrices in Chapter 4, sections 4.1-4.4. This will be helpful later in this section, and in the next section, and is good to review if you are rusty on linear algebra. (We will only need the very basics here.)
- Professor Eugene Lim at King's College London has some excellent lecture notes on Symmetry in Physics, focusing mostly on aspects of group theory, available at this link. I recommend especially sections 1 and 3. (Although note, his notes tend to be more mathematical than we will need.)
- Section 1 of Professor John McGreevy's Symmetry in Physics Lecture Notes from UCSD are quite good—available at this link. (Although, these notes also tend to be more mathematical than we will need.)

2.1 Symmetries as operations: geometric examples

To get started, let us return to the geometric examples we began with in the introduction. This will be helpful to establish some notation, and get a sense for how symmetry groups work, before we get into any formal definitions.

Example 1: Symmetries of an equilateral triangle For a first concrete example, consider an equilateral triangle. We will label the three corners by A, B, C as in Figure 2. What are the symmetries of the equilateral triangle? (Specifically, in this context we mean: *what are the rigid, distance-preserving motions of the plane that maps the triangle to itself*?)

For instance, intuitively if we rotate the equilateral triangle of 120°, the triangle remains the same. By "the same" we mean that if we drop the labels A, B, C, then you wouldn't be able to tell whether or not the triangle has been rotated. In such a case, we say that the triangle is *invariant* under such a rotation. This rotation is an example of an *operation* that we can perform on the triangle.

We can label the various symmetry operations of the equilateral triangle as follows:

- Doing nothing: this counts as a symmetry operation! It's called the *identity* operation, and for now we will denote it by the letter I. So by I, we mean *act on the equilateral triangle by the symmetry operation* I, which does nothing.
- Reflection about the vertical axis (through vertex A in the figure): S_v. Again, by S_v we mean *act on the equilateral triangle by the symmetry operation* S_v, which reflects it about the axis which goes through the vertex A.
- Reflection about the diagonal axis that goes through the left vertex (through vertex B in the figure): S_l
- Reflection about the diagonal axis that goes through the right vertex (through vertex C in the figure): S_r
- Rotation by 120° counterclockwise (taking $A \to B$, $B \to C$, $C \to A$ in the figure): R
- Rotation by 240° counterclockwise: since this operation is clearly achieved by two subsequent rotations by 120°, we'll denote it by $R \cdot R = R^2$. By this notation, we mean *act on the equilateral triangle by the symmetry operation R, and then subsequently act again by another symmetry operation R*, which composed together results in a total rotation by 240°. Note that rotation by a subsequent 120° (so by a total of 360°) gets us back to the original configuration, so is the same thing as the identity operation and not distinct.

Having gone through all of the possible reflectional and rotational symmetries of the equilateral triangle, are we missing any? To check explicitly, we should take the set of distinct operations we have identified so far,

$$\{I, S_v, S_l, S_r, R, R^2\} \tag{2.1}$$

and verify whether composing any of these together yields a new symmetry operation. For example, let's first rotate by 120°, and then reflect through the vertical axis S_v. We'll denote the composition of these two actions by $S_v \cdot R$, where our notation is that we act from right-to-left: first act by the rotation R, and then by the reflection S_v. This "backwards" *right-to-left* rather than *left-to-right* notation common in math is used since we imagine the action as being to the right on some object (in this case the triangle); you should have in mind

$$S_v \cdot R\,(\triangle) \quad \Rightarrow \quad \text{first } R(\triangle)\,, \text{ then } S_v(R(\triangle))$$

We can check that the result of this series of operations yields the same result as just the operation S_l, reflecting through the bottom left vertex; in other words,

$$S_v \cdot R = S_l$$

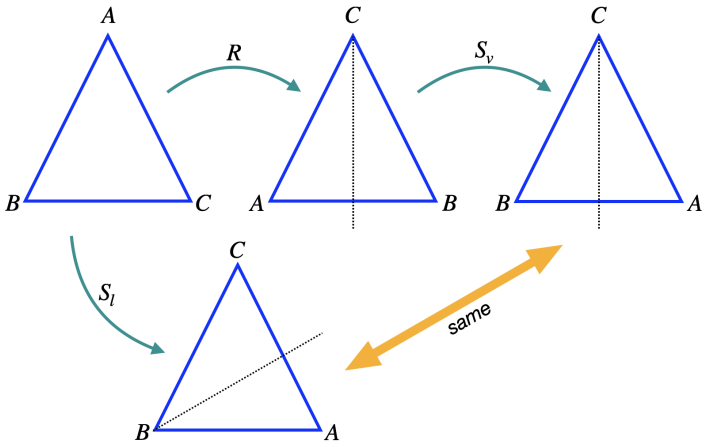


Figure 3: Demonstrating that $S_v \cdot R = S_l$.

and so this is not a distinct symmetry operation. In fact, you can check that any two compositions of the 6 distinct symmetry operations listed in (2.1) does not yield a new symmetry operation, so we have indeed captured everything.

Be careful, since the order in which we apply the operations matters! Applying first R and then S_v is *not* the same as applying first S_v and then R: in the latter case, we obtain

$$R \cdot S_v = S_r$$

In other words, the operations R and S_v do not *commute* with one another.

Exercise 2.1

Verify that we have correctly identified the 6 distinct symmetry operations of an equilateral triangle by filling out the following table of compositions:

$a \cdot b$	I	R	R^2	S_v	S_r	S_l
I						
R				S_r		
R^2						
S_v		S_l				
S_r						
S_l						

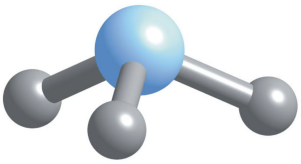
In this table (called a *Cayley table*), the rows refer to the elements a, and the columns to elements b, and the table entries are $a \cdot b$ (so, first act with the column element b, and then act with the row element a). For example, the entries $S_v \cdot R = S_l$ and $R \cdot S_v = S_r$ that we have already checked have been filled in. You should find that every entry of this table is one of the six elements (2.1).

You will notice that for every symmetry operation, there is another symmetry operation that undoes it. We call this the operation's *inverse*. The operation composed with its inverse gives back just the identity (*i.e.*, does nothing). For example, the inverse of a reflection is to just do the same reflection back — the mirror image of a mirror image is the original image. In particular, the inverse of S_v is S_v; the inverse of S_l is S_l, and the inverse of S_r is S_r. This is because composing $S_v \cdot S_v = I$, $S_l \cdot S_l = I$, $S_r \cdot S_r = I$.

Exercise 2.2

Find each of the inverses of the 6 operations $\{I, S_v, S_l, S_r, R, R^2\}$ for the equilateral triangle. In other words, for each of the operations O, which operator satisfies $O\cdot$(which operation) $= I$? Your list should only be composed of elements of $\{I, S_v, S_l, S_r, R, R^2\}$.

The NH_3 (ammonia) molecule is an example of an object that appears in nature whose symmetry operations provide a realization of the symmetry group of the equilateral triangle. The three H atoms form an equilateral triangle with the N atom lying on the symmetry axis that passes through the center of the triangle.



Example 2: Symmetries of a square Let us go through the same exercise for the symmetries of a square. The possible symmetry operations are as follows:

- Doing nothing: I.
- Rotation by 90 degrees counter-clockwise. Calling this transformation R, this generates 3 distinct symmetry transformations: R, $R \cdot R = R^2$ (rotation by 180 degrees counter-clockwise), $R \cdot R \cdot R = R^3$ (rotation by 270 degrees counter-clockwise).
- There are two symmetry transformations corresponding to reflection across each of the vertical and horizontal axes. Call these $S_{\leftrightarrow}$ (mirror reflection left-to-right across the vertical axis) and $S_{\updownarrow}$ (mirror reflection about the horizontal axis). Doing these twice just returns you to the original configuration, so $S^2_{\updownarrow}$ and $S^2_{\leftrightarrow}$ are not distinct transformations.
- There are additionally two reflections across each of the diagonal axes, from top left to bottom right (call it S_r), and from top right to bottom left (call it S_l).

In total, we have described the following set of 8 distinct transformations of the square:

$$\{I, R, R^2, R^3, S_{\leftrightarrow}, S_{\updownarrow}, S_l, S_r\} \tag{2.2}$$

You can verify for yourself that these are all the possible unique symmetry transformations.

Exercise 2.3

(a) Show that every element of the set (2.2) can actually be written as some composition of the following two actions: R, and $S_{\leftrightarrow}$. These are called the *generators* of the symmetry.

(b) Show that first acting with $S_{\leftrightarrow}$ and then with R is *not* the same thing as first acting with R and then with $S_{\leftrightarrow}$. Therefore, these transformations do not commute with one another ($R \cdot S_{\leftrightarrow} \neq S_{\leftrightarrow} \cdot R$).

(c) What is the inverse of $S_{\leftrightarrow}$? What about the inverse of R?

Example 3: Symmetries of a circle The previous examples we considered were examples of **discrete** symmetries: all of the transformations of the objects that described symmetries were indexed by discrete labels. (Reflection about an axis, which I can do only once or twice; rotation about a discrete number of units, in multiples of 120° for the equilateral triangle, versus in multiples of 90 for the square.)

This is different from a symmetry which is indexed by a continuous label, called a **continuous** symmetry. For example, consider the symmetries of a circle. A circle is invariant under rotations by infinitesimally small angles ϕ, as well as reflections about any axis through the origin. Let's focus on the rotations: these rotations can be labeled as $R(\phi)$, where ϕ is a continuous label for this set of symmetries, since it can be any real number. Without loss of generality, we can restrict ϕ to be some real number between 0 and 2π, since after rotation by 2π we get back to the same exact configuration we started with (the same thing as rotating by zero degrees) — in other words,

$$R(\phi + 2\pi) \simeq R(\phi)\,. \tag{2.3}$$

A bit more formally, we can describe the symmetries of a circle by labeling the coordinates of a given point on the circle by its polar coordinates, the angle θ and radius r. In complex coordinates, this point is labeled by $z = re^{i\theta}$, or in real Cartesian coordinates, we would label $(x = r\cos\theta, y = r\sin\theta)$. For simplicity, let's consider the unit circle, with radius $r = 1$. Then, a point is labeled uniquely by the real angle $0 \leq \theta < 2\pi$.

A symmetry transformation that rotates the circle by an angle ϕ, $R(\phi)$, takes $\theta \to \theta + \phi$. Composing two of these symmetry operations, say first by rotating by an angle ϕ_1 and then by an angle ϕ_2, is accomplished by acting on the coordinates as:

$$R(\phi_2)\cdot R(\phi_1): \quad e^{i\theta} \to e^{i(\theta+\phi_1)} \to e^{i(\theta+\phi_1+\phi_2)}$$

where of course, $e^{i(a+b)} = e^{ia}e^{ib}$. We can rotate all the way up to $\phi = 2\pi$ before we start double counting, so we should identify rotations $\phi \to \phi + 2\pi$ as being equivalent. In other words,

$$e^{i(\theta+2\pi)} = e^{i\theta} \quad \text{since} \quad e^{2\pi i} = 1\,.$$

Then, the equivalence relation (2.3) is naturally imposed.

You'll notice that unlike the previous two examples, which had 6 and 8 distinct symmetry operations, this example has an infinite number of symmetry transformations, since there are an infinite number of real numbers between 0 and 2π. (This is because ϕ that labels the symmetry transformation can be infinitesimally tiny!) This demonstrates the distinction between a **finite** symmetry and an **infinite** symmetry. The symmetry group of a circle is both *continuous* (labeled by a continuous parameter) and *infinite* (consists of an infinite number of distinct operations).[2]

——— *End Lecture 2.*

2.2 An introduction to finite groups

Now that we have some familiarity with some simple symmetry actions, let us more formally define what is meant by a symmetry group. Group theory is the branch of mathematics that defines abstract relationships among sets of elements in terms of operators among elements of a group.

[2] Note that the restriction (2.3) does not change the fact that the symmetry is infinite, since there are still an infinite number of real numbers between 0 and 2π.

Definition 2.1. A **group** G is a set of elements $\{g\}$ with a multiplication law (denoted by $\cdot$) that maps two group elements to a third group element.[3] The multiplication law has the following properties:

1. **Closure:** The product (composition) of any two group elements yields another member of the set. In particular, if g_1 and g_2 are elements of the group G, the product $g_1 \cdot g_2 = g_3$ yields an element g_3 which is also a member of G. This is called the *closure* property of the group, since the group "closes" under its multiplication law. The mathematical symbol for "is a member of a set" is $\in$, so we can write this rule as
$$g_1 \cdot g_2 = g_3\,, \qquad \text{for} \quad g_1, g_2, g_3 \ \in G$$
2. **Associativity:** The product is *associative*: $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$.
3. **Identity:** There is a an identity element, which we will call e, which satisfies: $e \cdot g = g \cdot e = g$ for all $g \in G$. This is the "do nothing" operation.
4. **Inverse:** Every element g has a unique inverse element g^{-1}, such that $g^{-1} \cdot g = e$, and $g \cdot g^{-1} = e$.

Any set of elements with a composition/multiplication rule which obeys all of the above properties is called a group. An example of a group is the group of symmetries of the equilateral triangle, which we have already discussed. This symmetry group is called the *dihedral group* D_3. In general, the dihedral group D_n is the group of symmetries of a regular polygon with n sides. D_3 is a symmetry group consisting of 6 distinct elements, which in the previous section we called,
$$g = \{I, R, R^2, S_v, S_l, S_r\} \tag{2.4}$$
In Exercise 2.2 you explored the composition of these elements, and should have found the following multiplication table:

$a \cdot b$	I	R	R^2	S_v	S_r	S_l
I	I	R	R^2	S_v	S_r	S_l
R	R	R^2	I	S_r	S_l	S_v
R^2	R^2	I	R	S_l	S_v	S_r
S_v	S_v	S_l	S_r	I	R^2	R
S_r	S_r	S_v	S_l	R	I	R^2
S_l	S_l	S_r	S_v	R^2	R	I

Running down the list of group properties, we can verify that each is true for D_3:

1. **Closure:** Composing any two of the elements (2.4) leads to a third element in this list. This is precisely what you were asked to verify in Exercise 2.1, when you showed that every member of the Cayley table can be written as one of these 6 elements.
2. **Associativity:** This property can be evaluated with the multiplication table. For example, consider associativity of the product:
$$S_v \cdot (S_r \cdot S_l) \stackrel{?}{=} (S_v \cdot S_r) \cdot S_l$$

[3] We will interchangebly use the terms *group multiplication law* and *group composition law*. Here we understand "multiplication" to be a composition rule that satisfies properties (1)-(4), not literal multiplication of numbers!

On the left-hand-side of the equals sign, we are first asked to evaluate $S_r \cdot S_l = R^2$, and then to evaluate $S_v \cdot R^2 = S_r$. On the right-hand-side, we are first asked to evaluate $S_v \cdot S_r = R^2$, and then to evaluate $R^2 \cdot S_l = S_r$. Either way, we get the same answer S_r, so this product is associative. One can verify that this associativity property holds true for all other possible three-way groupings of the elements (2.4).

3. **Identity:** Previously we called the identity element $e = I$, and explicitly verified via the table that $I \cdot (\text{any element } g) = g$, and $(\text{any element } g) \cdot I = g$. This is reflected by the first row and first column of the table.

4. **Inverse:** The inverse of an element is the unique element that multiplies it to give the identity—in other words, the action that undoes the previous action. You showed in Exercise 2.2 that each element (2.4) has an inverse which is also a member of (2.4), and since I only appears once in each row and column, the inverse is unique.

Some further remarks are in order:

- The order of operations might matter! It is not necessarily true that $g_1 \cdot g_2 = g_2 \cdot g_1$. (In particular, associativity does *not* imply commutivity.) If all products commute, so that ordering doesn't matter, the group is called **abelian**:

$$G \text{ abelian:} \quad [g_1, g_2] = 0 \quad \text{for all} \quad g_1, g_2 \in G$$

  Otherwise, it is called **non-abelian**. You've already seen that the triangle group D_3 is non-abelian, since (for example) $S_v \cdot R \neq R \cdot S_v$. It turns out that D_3 is the smallest non-abelian group, in the sense that it is the non-abelian group with the smallest number of distinct elements.

- Another definition: the number of distinct elements of the group is called the **order** of the group. The order might be finite, with some finite number of elements (like D_3, which has order 6), or infinite (like the symmetry group of the circle). Furthermore, the group elements might be labeled by discrete variables (like integers), or continuous variables (like real numbers, as we saw in the example of the circle). D_3 is an example of a finite, discrete, non-abelian group.

- A finite group can be completely specified by its multiplication table, called a **Cayley table**. We have already seen an example of a Cayley table for the symmetry group of the equilateral triangle. This Cayley table actually contains *all* of the information about D_3.

The other example we considered in the previous section was the symmetry group of the square. This is a finite, discrete, non-abelian group of order 8, called the dihedral group D_4. Recall that in that case, we called attention to the fact that while the set of 8 operations (2.2) are the complete set of distinct symmetry operations, some of them can actually be written as compositions as other group elements. In particular, in Exercise 2.3 you should have found that every element can actually be written as the composition of only two elements: $R, S_{\leftrightarrow}$. In general, we can ask the question: *what is the minimal number of operators needed to generate the entire set of distinct group elements?* These minimal subset of operators are called the **generators**. For example, we would say that "the symmetry group D_4 is generated by $\{R, S_{\leftrightarrow}\}$," since every other element can be represented by these two. As we will see in examples later, group generators play a crucial role in understanding symmetries in physics.

Exercise 2.4

(a) For a finite symmetry group, what property must the Cayley table satisfy if the group is abelian?

(b) *The Sudoku rule:* Show that every group element must appear exactly once in each row and in each column of the Cayley table. In other words, show that $g_1 \cdot g_3 = g_2 \cdot g_3$ implies $g_1 = g_2$. (You will only need to use the 4 group properties to show this.)

Example: the integers under addition To contrast with the finite, geometric examples we have studied thus far in this subsection, let's consider a different example to round out this discussion. The set of all integers $\{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$ form a group under the composition law of addition. In particular, we mean that the set of all integers $\{-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty\}$ (often denoted by $\mathbb{Z}$) is a group if we consider the group multiplication law to correspond to the operation of adding integers:

$$g_2 \cdot g_2 \equiv g_1 + g_2 \quad \text{for } g_1, g_2 \in \mathbb{Z}$$

(The symbol $\equiv$ means *defined as.*) We can verify this statement by running down the list of four group properties. Closure clearly holds, since adding two integers always gives a third integer. Furthermore, addition is clearly associative: $(g_1 + g_2) + g_3 = g_1 + (g_2 + g_3)$. The identity element is identified with 0, and indeed $g + 0 = 0 + g = g$ for g any integer. And furthermore, the inverse of every integer is its negative, which is still a member of $\mathbb{Z}$: $g^{-1} = -g$, since $g + (-g) = (-g) + g = 0$. This group is:

- *abelian*, since addition is always commutative, $g_1 + g_2 = g_2 + g_1$ for any two integers;
- *discrete*, since it is labeled by discrete integers;
- *infinite*, since there are an infinite number of integers.

Exercise 2.5

(a) Fill out a portion of the Cayley table for the integers under addition for the subset $g = \{-2, -1, 0, 1, 2, 3\}$. (This will be a 6×6 table, which is some finite portion of the infinite Cayley table for this infinite group.)

(b) Is the set of integers a group under the composition law of multiplication, rather than addition? Why or why not?

Now that we have a feel for what a group is, in subsequent sections we will explore a variety of other examples of groups that exemplify various group properties.

2.3 The cyclic group Z_2 and a first pass at subgroups, isomorphisms, and representations

Let's go back to the symmetry group of the equilateral triangle, D_3. We saw that reflection across a symmetry axis — for example, S_v implementing reflection across the vertical axis — is a symmetry operation which is its own inverse,

$$S_v \cdot S_v = I \,.$$

In fact, if we restrict to just the two elements $\{I, S_v\}$, these two elements alone form a group of order 2, which we'll call Z_2. We can verify that all 4 group properties hold for Z_2:

1. The group Z_2 closes: filling out the multiplication table,

$a \cdot b$	I	S_v
I	I	S_v
S_v	S_v	I

   so that the composition of the two elements in any order always gives back one of those same two group elements.

2. The product is associative. (If this was true for the larger group D_3, then it's certainly true for this order 2 subgroup.)

3. The identity $e = I$ is in Z_2.

4. The inverses of I and S_v are both also in Z_2, since the inverse of I is I itself, and the inverse of S_v is S_v itself.

Z_2 is an example of a *subgroup* of D_3: a subset of a group that itself satisfies all the conditions to be its own group. A little more formally:

Definition 2.2. Let G be a group. A subset H of G is a **subgroup** of G if:[4]

- H closes: For every $h_1, h_2 \in H$, the product $h_1 \cdot h_2 \in H$. (Of course, since the elements of H are also elements of G, h_1, h_2 are also elements of G.)
- The identity of G is in H.
- If h is in H, then its inverse h^{-1} is also in H.

We have seen that there is an order 2 subgroup Z_2 of the order 6 group D_3 which corresponds to reflections about the vertical symmetry axis. This smaller group Z_2 is generated by just the single element S_v, since the identity I can be written in terms of S_v as $S_v \cdot S_v = I$, whereas the larger group D_3 has two generators. It must be stressed, however, that this is just *one* manifestation of the abstract group that we have called Z_2. First of all, D_3 has other Z_2 subgroups: for example, the subgroup consisting of $\{I, S_l\}$ satisfies the exact same multiplication table as $\{I, S_v\}$, and so provides another manifestation of the group Z_2.

Furthermore, while in these examples we have identified the two Z_2 elements with operations that act on an equilateral triangle, the triangle is <u>not</u> necessary to study/describe the underlying group Z_2. We can drop the triangle entirely, and just study the properties of 2 elements (call them e and b) that possess the algebraic properties described above,

$$Z_2 = \{e, b\}\,, \qquad e \cdot e = b \cdot b = e\,, \quad e \cdot b = b \cdot e = b$$

Before we were thinking of $e = I$ as the operation that does nothing to the equilateral triangle, and $b = S_v$ as the operation that reflects the equilateral triangle across its vertical axis, but we don't need to mention the equilateral triangle at all.

[4] As an aside.. any group has two trivial subgroups: itself (of course), and the trivial subgroup consisting of nothing but the identity of G, $\{e\}$. Typically when we discuss subgroups of a group, we exclude these two obvious ones.

Exercise 2.6

Show that the composition law $e \cdot b = b \cdot e$ for the group Z_2 can be obtained from the relation $b \cdot b = e$ satisfied by the generator b.

This abstraction is a crucial part of the study of group theory, and the reason why group theory is so powerful in describing symmetries! Once we know all the properties of a group, then any physical system that possesses the same symmetries will obey exactly the same underlying group-theoretic properties. For instance, it turns out that *any* finite symmetry group of order 2 is equivalent to the group Z_2, since Z_2 is the unique group of order 2. Therefore, if I find an order 2 finite group in my system, I know that it is Z_2. For this reason, despite being the simplest possible nontrivial group, Z_2 is in fact one of the most important groups in physics. Consider the following example.

Another realization of Z_2 Suppose we have a function of space $\psi(x)$, for $-\infty < x < \infty$. (For example, you might have in mind the wavefunction in a 1d quantum mechanical system.) Consider the operation of sending $x \to -x$, *i.e.* flipping the argument of $\psi(x)$. This operation is called **parity**, so we will denote it with a letter P:

$$P: \quad \psi(x) \to \psi(-x) \quad \leftrightarrow \quad P\psi(x) = \psi(-x)$$

By the latter notation, we mean "the operator P acting on $\psi(x)$ yields $\psi(-x)$." Consider also the identity operation of "doing nothing" to the wavefunction,

$$e: \quad \psi(x) \to \psi(x) \quad \leftrightarrow \quad e\psi(x) = \psi(x)$$

Clearly, acting on $\psi(x)$ twice with the parity operation just gives back the original function:

$$P \cdot P\psi(x) = P\psi(-x) = \psi(x)$$

These two operators $\{e, P\}$ then form a group of order 2; dropping the $\psi(x)$, we have just shown that these two elements satisfy the multiplication laws $eP = Pe = P$, and $e^2 = P^2 = e$. We recognize these as precisely the group multiplication laws of the cyclic group Z_2. Parity (defined in higher dimensions as the reflection of the spatial coordinates about the origin of space, $\vec{x} \to -\vec{x}$) is an example of a Z_2 group satisfied by many quantum systems.[5]

The notion of two groups being the "same" is formalized by the idea of *isomorphism*. Intuitively, even though two groups G and G' might have elements that look different, they are said to be the same, or isomorphic, if their composition laws have the same structure. Formally,

Definition 2.3. Two groups G and G' are said to be **isomorphic**, $G \cong G'$, if there exists a map between their group elements that preserves the group multiplication law, and which is bijective (one-to-one and onto).

We have just seen that the cyclic group Z_2 is isomorphic to parity symmetry.

——— *End Lecture 3.*

[5] Not just quantum systems! Newton's second law is also invariant under parity! Newton's second law is of the form $\vec{F} = m\vec{a}$, so that under the transformation $\vec{x} \to -\vec{x}$ both vectors on the left and right side of this equation get the same minus sign.

A historical aside on parity symmetry: Until 1956 it was assumed that the interactions of the fundamental particles were invariant under parity. Based on experiments involving subatomic particles called kaons, in 1956 physicists Tsung-Dao Lee and Chen Ning Yang proposed that parity does *not* apply to the weak force, the force responsible for radioactive decay of subatomic particles. This prediction was quickly verified by Chien-Shiung Wu in 1957, earning Lee and Yang the Nobel Prize in physics.

Action of operators on vector spaces We have just seen two realizations of the group Z_2: one in which the group elements act on an equilateral triangle, and one in which the group elements act on a function of space. In either case we had in mind the following kind of action:

$$operator \times \text{object} = \text{result} \tag{2.5}$$

where the group element (acting as an "operator") acted on an object (the a point on the triangle, or the argument of the function), which did something to the object (yielding the "result"). More generally, it is often extremely useful to mathematically describe the "object" without referencing any specific concrete realization of the symmetry; for example, without referencing the triangle. How to do this?

In these physics, we will often be interested in the case where the object lives in a *vector space*. To motivate this idea, let's go back to the equilateral triangle. Any point (x, y) on the triangle can be described by a vector in the x-y plane pointed from the origin to (x, y),

$$\vec{v} = x\hat{x} + y\hat{y}$$

where $\hat{x}$ and $\hat{y}$ are the unit vectors pointing in the x and y directions, respectively. Choosing to place the origin of our coordinate system at the center of the triangle, we can label the three triangle vertices as being at points (x, y) (or equivalently, vectors $\vec{v}$) equal to

$$\begin{aligned} A = (0,1) &\quad\leftrightarrow\quad \vec{A} = \hat{y}\,, \\ B = (-\frac{\sqrt{3}}{2}, -\frac{1}{2}) &\quad\leftrightarrow\quad \vec{B} = -\frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y}\,, \\ C = (\frac{\sqrt{3}}{2}, -\frac{1}{2}) &\quad\leftrightarrow\quad \vec{C} = \frac{\sqrt{3}}{2}\hat{x} - \frac{1}{2}\hat{y} \end{aligned}$$

Actually, it will be convenient to introduce a matrix notation, where we represent the unit vectors as unit column vectors,

$$\hat{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \hat{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These two vectors form a basis of a 2-dimensional vector space. A general vector $\vec{v}$ pointing from the origin to the point (x, y) is given in this basis by the column vector

$$\vec{v} = x \begin{pmatrix} 1 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

For example, in this matrix notation the three vertices occur at locations

$$\vec{A} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{B} = \begin{pmatrix} -\sqrt{3}/2 \\ -1/2 \end{pmatrix}, \qquad \vec{C} = \begin{pmatrix} \sqrt{3}/2 \\ -1/2 \end{pmatrix}$$

So far so good.

Next, we would like to represent the symmetry operators of D_3 in such a way that when they act on a vector (describing a point on the triangle), the result is another vector (describing another point on the triangle), according to how that operation transforms the triangle. It is clear that the symbol $\times$ in (2.5) should then denote *matrix multiplication*. In order to for the "result" of (2.5) to be a valid vector, the *operators* (call them D) must be 2×2 matrices of the following form,

$$\textit{operator } D = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad D \times \vec{v} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax + by \\ cx + dy \end{pmatrix} = \vec{v}'$$

Here a, b, c, d are real numbers.

Now that we have represented the object (the space in which the 2d equilateral triangle lives) in terms of 2-component vectors, and the symmetry operations (group elements) by 2×2 matrices, we can figure out: what are the matrices that represent the D_3 group elements? For simplicity, let's start with the Z_2 subgroup $\{I, S_v\}$. The identity element, $e = I$ should take any vector $\vec{v}$ back to itself, so of course should be represented by the identity matrix:

$$D(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We are using a somewhat pedantic notation where $D(I)$ means "the matrix representing the group element I", although later we will often just drop the "$D(I)$" and write $I = \ldots$. Then, you can check explicitly that $D(I)\vec{v} = \vec{v}$:

$$D(I)\vec{v} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} = \vec{v} \quad \checkmark$$

What about the element S_v that reflects about the central vertical axis? In our vector space this is a reflection about the y-axis, which takes all x-coordinates to minus themselves, and leaves the y-coordinates untouched. This is accomplished with:

$$D(S_v) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} : \qquad D(S_v)\vec{v} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}$$

In particular, we can verify that acting with S_v on each of the 3 vertices yields,

$$D(S_v)\vec{A} = \vec{A}\,, \qquad D(S_v)\vec{B} = \vec{C}\,, \qquad D(S_v)\vec{C} = \vec{B}$$

which flips the B and C vertices, exactly as the reflection S_v should.

With this matrix representation of the group elements, the group composition law $(\cdot)$ is also described by matrix multiplication. We can straightforwardly verify that we reproduce the Z_2 Cayley table using matrix multiplication:

$$\begin{aligned} I^2 = I \quad &\leftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark \\ S_v \cdot S_v = I \quad &\leftrightarrow \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark \\ I \cdot S_v = S_v \quad &\leftrightarrow \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark \\ S_v \cdot I = S_v \quad &\leftrightarrow \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark \end{aligned} \tag{2.6}$$

To summarize: we have accomplished representing the Z_2 group elements by matrices acting on a 2-dimensional (real) vector space. These two matrices $D(I)$ and $D(S_v)$ (along with the 2d vector space represented by the unit column vectors $\hat{x}$, $\hat{y}$) form what is called a **group representation** of Z_2. Of course, the vertices of the triangle clearly belong to this vector space, but they are just a subset of the infinitely many points in the vector space; the matrix operators we have constructed will act on any vector in this vector space.

This idea of group representations is extremely useful; arguably representation theory is the main point of contact between group theory and quantum physics (where the vector space in question is the Hilbert space of the physical system). Furthermore, by studying group representations, we can study groups using linear algebra, which allows for all the nice structure of linear algebra to be brought to bear in the study of symmetries. We are *not* going to delve into representation theory in these notes (this is a whole course in itself!), but we will be using matrix representations to describe symmetry groups in a variety of contexts.

Exercise 2.7

(a) Represent all 6 elements of D_3 in a 2×2 matrix representation.

(b) Identify three other subgroups of D_3 besides the Z_2 generated by S_v.

Yet another realization of Z_2 In general, group representations are *not* unique — there might be several or many different ways to represent the same group elements. And of course, we have already emphasize that the same group might show up in many different contexts. To really drive home these points, let's consider a completely different realization of the cyclic group Z_2. (As we said, this is a very important group!)

In computing, information is encoded by bits, which are logical states with one of two possible values. We can represent these two possible values by $\uparrow$ (or true, or 1) and $\downarrow$ (or false, or 0). The "NOT" operator (call it P) flips $\uparrow$ to $\downarrow$ and $\downarrow$ to $\uparrow$, *i.e.*

$$P\uparrow=\downarrow, \qquad P\downarrow=\uparrow$$

We of course also have the "do-nothing", or identity, operator, which leaves a bit as it is:

$$e\downarrow=\downarrow, \qquad e\uparrow=\uparrow$$

Flipping a bit twice gets you back to the original bit, $P \cdot P = e$, and so P is its own inverse. Associativity is also easily checked. So, the operators $\{P, e\}$ form an order 2 group, which is isomorphic to Z_2.

A group representation of these actions can be formulated as follows. We can represent the two possible states of the bit by the column vectors,

$$\uparrow=\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \downarrow=\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which span a 2d vector space. Then, the operators P and e can be represented by the following 2×2 matrices,

$$D(P) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad D(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.7}$$

You can verify via matrix multiplication that indeed, $D(P)^2 = e$, $D(e)^2 = e$, and $D(e)D(P) = D(P)D(e) = D(P)$. The matrices (2.7) therefore form a group representation of Z_2. However, this is clearly a *different* representation than the one we formulated in (2.6) — there the analogue of the element P was represented by a different 2×2 matrix! — but nonetheless, this is a good representation all the same. This example demonstrates the point that for any group there may exist more than one group representation. In this case, there is more than one distinct 2-dimensional representation (meaning, representations by 2×2 matrices).

——— *End Lecture 4.*

2.4 More finite examples: The cyclic group Z_n and symmetric group S_n

Let us expand our study to groups of higher order. The order 2 group Z_2 is a member of a larger class of groups Z_n called the *cyclic group* (also often denoted as C_n, although we will use the notation Z_n). The cyclic group is an order n group generated by a single element, which we will call b. Its n total elements can be expressed by composing b with itself n times:

$$Z_n = \{b,\, b^2, b^3,\, \ldots,\, b^{n-1}, b^n = e\}$$

where we have denoted $b \cdot b$ by b^2, $b \cdot b \cdot b$ by b^3, *etc.* (We emphasize again that the group composition law $(\cdot)$ does *not* necessarily correspond to regular multiplication of the elements — as we have seen in many examples already this will depend on how we represent the group action!) For example, Z_2 is generated by the single element b satisfying $b^2 = e$, where in the context of the equilateral triangle we called $b = S_v$, and in the context of parity we called $b = P$.

As you might have guessed, there are several equivalent realizations of the group Z_n. We'll give two examples below.

(1) Integers modulo n under addition A presentation of the cyclic group which you might have seen before is as follows. Let b and n be positive integers. The expression b mod n means to "take the remainder when b is divided by n." For example:

$$\begin{aligned}
32 \bmod 5 = 2 \quad &\text{since} \quad \frac{32}{5} = 6 + \frac{2}{5} \quad \leftrightarrow \quad 32 = 5 \times 6 + 2 \\
14 \bmod 8 = 6 \quad &\text{since} \quad \frac{14}{8} = 1 + \frac{6}{8} \quad \leftrightarrow \quad 14 = 1 \times 8 + 6 \\
15 \bmod 3 = 0 \quad &\text{since} \quad \frac{15}{3} = 5 + \frac{0}{3} \quad \leftrightarrow \quad 15 = 5 \times 3 + 0
\end{aligned}$$

Much like the integers under addition form an (infinite) group, the integers *modulo* n under addition form a (finite) group, where the composition law between two integers a, b is defined

$$a \cdot b = (a + b) \bmod n\,.$$

and the identity is $e = 0$. This group is often referred to as $\mathbb{Z}_n$.

For example, consider the integers modulo 2, $\mathbb{Z}_2$. Suppose we wish to form the Cayley table

for a subset of the integers, $-2, -1, 0, 1, 2, 3$ for this group. For example, we can compute

$$\begin{aligned}
\text{“}(-2)\cdot(-2)\text{”} &= (-2-2) \bmod 2 = -4 \bmod 2 = 0 \\
\text{“}(-2)\cdot(-1)\text{”} &= (-2-1) \bmod 2 = -3 \bmod 2 = 1 \\
\text{“}(-2)\cdot 0\text{”} &= (-2+0) \bmod 2 = -2 \bmod 2 = 0 \\
\text{“}(-2)\cdot 1\text{”} &= (-2+1) \bmod 2 = -1 \bmod 2 = 1 \\
\text{“}(-2)\cdot 2\text{”} &= (-2+2) \bmod 2 = 0 \bmod 2 = 0 \\
\text{“}(-2)\cdot 3\text{”} &= (-2+3) \bmod 2 = 1 \bmod 2 = 1
\end{aligned}$$

and so on for the other rows of the table. (More explicitly, for instance $-3 \bmod 2$ is computed by $\frac{-3}{2} = -2 + \frac{1}{2}$, *etc.*) What you'll find is that the answer is always either 0 or 1. In fact, *every* integer modulo 2 is equivalent to either 0 or 1, so this group consists of only those two elements. Said more precisely, the operation mod 2 defines an equivalence class on the integers which equates every integer with either 0 or 1. For instance,

$$\begin{aligned}
-2 \bmod 2 = 0\,, \qquad -1 \bmod 2 = 1\,, \qquad 0 \bmod 2 = 0 \\
1 \bmod 2 = 1\,, \qquad 2 \bmod 2 = 0\,, \qquad 3 \bmod 2 = 1
\end{aligned}$$

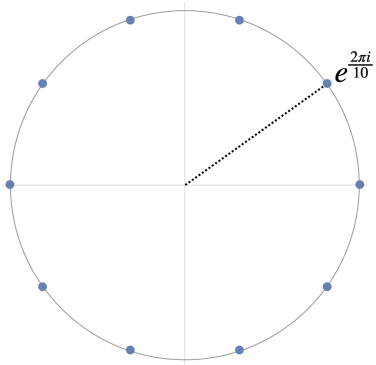
and so on. The Cayley table for this order 2 group can be expressed as,

$a \cdot b =$ $(a+b) \bmod 2$	0	1
0	0	1
1	1	0

Clearly, this group can be considered as generated by the single element 1, since $1 \cdot 1 = 0$ (the identity element). Having reproduced the Cayley table for $\mathbb{Z}_2$, we see that $\mathbb{Z}_2 \cong Z_2$. For more general n, the group $\mathbb{Z}_n$ is isomorphic to the cyclic group Z_n, $\mathbb{Z}_n \cong Z_n$.

Exercise 2.8

The group Z_5 is order 5, and equivalent to the integers under addition modulo 5 ($\mathbb{Z}_5$). Fill out the Cayley table for this group, using representative elements $\{0, 1, 2, 3, 4\}$. Show that this group has 1 generator.



(2) The nth roots of unity Another presentation of the cyclic group Z_n is to let the elements b correspond to nth roots of unity, $b = e^{2\pi i/n}$, with the group multiplication law corresponding

to multiplication of these roots:

$$b = e^{2\pi i/n}\,, \qquad b \cdot b = e^{2\pi i/n} e^{2\pi i/n} = e^{4\pi i/n}\,.$$

The example of $n = 10$ is shown in the figure. The roots of unity naturally satisfy $b^n = e$, since

$$(e^{2\pi i/n})^n = e^{2\pi i} = 1 = e$$

while the other products fill out the group elements,

$$\{e^{2\pi i/n}\,,\ e^{4\pi i/n}\,, \ldots\,,\ e^{(n-1)2\pi i/n}\,, 1\}$$

Exercise 2.9

The set of complex numbers $\{1, a, a^2\}$ for $a = e^{2\pi i/3}$ under multiplication form the cyclic group Z_3. Find a representation of this group in terms of 2×2 complex matrices. (In other words, find three 2×2 matrices $D(1)$, $D(a)$, $D(a^2)$ that satisfy the group multiplication table, $D(a)^2 = D(a^2)$, $D(a)^3 = D(1)$, $D(a)D(e) = D(e)D(a) = D(1)$, *etc.*)

The symmetric group S_n Suppose you are given n balls of different colors, and are asked to put them into n separate boxes. How many possible ways can you do that? For example, given 3 balls of different colors (red, blue, green), there are 6 possible permutations:

Box 1	Box 2	Box 3
R	B	G
R	G	B
B	R	G
B	G	R
G	R	B
G	B	R

In general, the set of all permutations of n objects forms a group S_n, called the *symmetric group*. This is a group of order $n!$, since in general there are $n!$ such permutations of n distinct objects.

For example, consider the group S_3. We may represent the action of all the possible permutations of 3 objects by how such permutations act on the list of numbers (1 2 3). The $n!$ possibilities would take (1 2 3) to: (1 2 3) (doing nothing); (1 3 2); (2 1 3); (2 3 1); (3 2 1); (3 1 2). A nice way to write these 6 symmetry operations is with the following notation,

$$P = \begin{pmatrix} 1 & 2 & 3 \\ p_1 & p_2 & p_3 \end{pmatrix} \equiv \text{ take } 1 \to p_1,\ 2 \to p_2,\ 3 \to p_3$$

where p_1, p_2, p_3 are some rearrangement of the numbers $1, 2, 3$. Thus, we can write the inverse of P as

$$P^{-1} = \begin{pmatrix} p_1 & p_2 & p_3 \\ 1 & 2 & 3 \end{pmatrix} = \text{ take } p_1 \to 1,\ p_2 \to 2,\ p_3 \to 3$$

so that successively applying P and then P^{-1} returns the numbers to their original configuration. (This is NOT a matrix group representation, just a notation for labeling different permutations of 3 numbers.)

With this notation, the 6 possible permutations can be written as,

$$P_0 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \qquad P_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$$
$$P_3 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \qquad P_4 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \qquad P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

To obtain the permutation from applying two subsequent permutations, for example

$$P_1 \cdot P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

(by which we mean first act with P_5, and then with P_3), we should rearrange the columns of the left matrix so that the top row matches the lower row of the right matrix, and then reduce to a matrix with the resulting bottom left row under the top right row,

$$P_1 \cdot P_5 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = P_3$$

Try it out a couple of times and you'll get the hang of it. Going through this exercise for all the permutations, you should find the following table:

$a \cdot b$	P_0	P_1	P_2	P_3	P_4	P_5
P_0	P_0	P_1	P_2	P_3	P_4	P_5
P_1	P_1	P_2	P_0	P_4	P_5	P_3
P_2	P_2	P_0	P_1	P_5	P_3	P_4
P_3	P_3	P_5	P_4	P_0	P_2	P_1
P_4	P_4	P_3	P_5	P_1	P_0	P_2
P_5	P_5	P_4	P_3	P_2	P_1	P_0

Does this table look familiar? If I call

$$P_0 = I\,, \quad P_1 = R\,, \quad P_2 = R^2\,, \quad P_3 = S_v\,, \quad P_4 = S_r\,, \quad P_5 = S_l$$

Then this is *exactly* the same table that we wrote previously for the dihedral group D_3, the symmetry group of the equilateral triangle (written below (2.4)). Indeed, we can understand the elements of this group as the various permuted indices of the equilateral triangle labeled $(A = 1, B = 2, C = 3)$, and the group composition as the act of permuting the vertices – for instance, P_1 takes $A \to B$, $B \to C$, $C \to A$, indeed corresponding to a rotation of the triangle. Since a finite group is defined by its multiplication table, these two groups – S_3 and D_3 – are one and the same, and the group S_3 is isomorphic to D_3.

2.5 Rotations in 2d and 3d

A rotation is a linear transformation that leaves the lengths of vectors unchanged. For instance, consider the group of rotations $0 \leq \theta < 2\pi$ about some axis in the 2d plane. You will recall that around (2.3) we denoted these group elements by $R(\theta)$, which can be written as complex phases; here let's use the notation g_θ to denote the group elements,

$$g_\theta = \{e^{i\theta}\,,\ 0 \leq \theta < 2\pi\}$$

This is a continuous group, as it is labeled by a continuous real variable $\theta \in \mathbb{R}$. As you will also recall, this is part of the symmetry group of the circle (which also contains the set of continuous reflection symmetries).

Group multiplication corresponds to the multiplication of the phases, which adds the angles:

$$g_{\theta_1} \cdot g_{\theta_2} = e^{i\theta_1} e^{i\theta_2} = e^{i(\theta_1+\theta_2)} = g_{\theta_1+\theta_2}$$

This composition is also clearly commutative: $e^{i\theta_1} \cdot e^{i\theta_2} = e^{i\theta_2} \cdot e^{i\theta_1}$ since $\theta_1 + \theta_2 = \theta_2 + \theta_1$, so this is an abelian group. The identity element is just $e^0 = 1$, and inverses are given by minus the angle:

$$(g_\theta)^{-1} = g_{-\theta} = g_{2\pi-\theta} \quad \Rightarrow \quad (g_\theta)^{-1} g_\theta = g_{2\pi-\theta+\theta} = 1 \quad \checkmark$$

(Recall that two angles θ and $\theta + 2\pi$ are equivalent, so in order to define the inverse angle in the range $[0, 2\pi]$ we should technically add the 2π.)

What about the representations of this group? We can define a 1-dimensional "vector", *i.e.* another complex number $v = re^{i\alpha}$ for r and α real positive numbers, $0 \leq r < \infty$ and $0 \leq \alpha < 2\pi$. Then, the action of the group elements g_θ on complex numbers v is simply to add θ to the angle α:

$$g_\theta v = re^{i(\alpha+\theta)}$$

In other words, it rotates v clockwise by the angle θ (as expected! We are just putting fancy words to things you already know about rotations). This defines a 1-dimensional representation.

Of course, it is also clear that the lengths of vectors are kept invariant under rotations, since pure complex phases always have length 1:

$$|g_\theta v| = \sqrt{(g_\theta v)(g_\theta v)^*} = r = |v| \quad \checkmark$$

You might have seen another way to represent rotations in 2-dimensional space. Instead of using complex variables, we can label the plane by real variables x and y. A rotation by angle ϕ takes a vector $(x, y) \to (x', y')$ as,

$$x \to x\cos\theta - y\sin\theta\,, \qquad y \to x\sin\theta + y\cos\theta$$

Such a transformation indeed preserves the length of the vector $\vec{v} = x\hat{x} + y\hat{y}$, since

$$|v'|^2 = x'^2 + y'^2 = (x\cos\theta - y\sin\theta)^2 + (x\sin\theta + y\cos\theta)^2 = x^2 + y^2 = |v|^2$$

This operation can be represented in matrix form by the following 2×2 matrix,

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{2.8}$$

which acts on a vector (x, y) as

$$R_\theta \vec{v} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}$$

——— *End Lecture 5.*

For example, taking $\theta = 120^\circ$ and 240° reproduces the 2-dimensional representations of R and R^2 in the group D_3, as you should have found in Exercise 2.7.

We can again verify that two consecutive rotations by angles θ_1 and θ_2 yield another rotation by angle $\theta_1 + \theta_2$; explicitly,

$$\begin{aligned} R_{\theta_2} \cdot R_{\theta_1} \vec{v} &= \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} x\cos\theta_1 - y\sin\theta_1 \\ x\sin\theta_1 + y\cos\theta_1 \end{pmatrix} \\ &= \begin{pmatrix} \cos\theta_2(x\cos\theta_1 - y\sin\theta_1) - \sin\theta_2(x\sin\theta_1 + y\cos\theta_1) \\ \sin\theta_2(x\cos\theta_1 - y\sin\theta_1) + \cos\theta_2(x\sin\theta_1 + y\cos\theta_1) \end{pmatrix} \\ &= \begin{pmatrix} x\cos(\theta_1+\theta_2) - y\sin(\theta_1+\theta_2) \\ x\sin(\theta_1+\theta_2) + y\cos(\theta_1+\theta_2) \end{pmatrix} = R_{\theta_1+\theta_2}\vec{v} \end{aligned}$$

where we used the trig identities

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1+\theta_2)\,, \quad \cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2 = \sin(\theta_1+\theta_2)\,.$$

Of course, there is a clear map between these two different ways of representing the 2d rotation group, since $\mathrm{Re}(e^{i\theta}) = \cos\theta$ and $\mathrm{Im}(e^{i\theta}) = \sin\theta$,

$$e^{i\theta} \to \begin{pmatrix} \cos\theta = \mathrm{Re}(e^{i\theta}) & -\sin\theta = -\mathrm{Im}(e^{i\theta}) \\ \sin\theta = \mathrm{Im}(e^{i\theta}) & \cos\theta = \mathrm{Re}(e^{i\theta}) \end{pmatrix}. \tag{2.9}$$

Rotations in 3d This discussion readily generalizes to three spatial dimensions. Consider the set of rotations in 3d space, which rotate a vector $\vec{x} = (x, y, z)$ to another vector $\vec{x}' = (x', y', z')$,

$$R\vec{x} = \vec{x}'$$

We can represent the group elements as 3×3 matrices R, so that

$$R \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}$$

A general 3d rotation matrix is obtained by composing rotations about each of the 3 axes. A rotation by angle α around the x-axis (so, in the y-z plane), β around the y-axis (in the x-z plane), and γ around the z-axis (in the x-y plane) is accomplished by appropriately embedding the 2d rotation matrix (2.8) into a 3d matrix, as

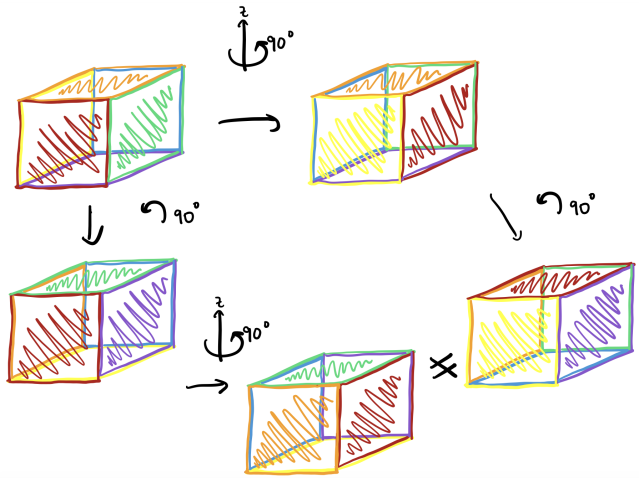
$$R_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}, \; R_y(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix}, \; R_z(\gamma) = \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

A general 3d rotation R can be specified by these three angles of rotation, one about each axis, and so is given by the composition of these three basis rotations,

$$\begin{aligned} R &= R_z(\gamma)R_y(\beta)R_x(\alpha) \\ &= \begin{pmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \end{aligned} \tag{2.10}$$

This group is non-abelian – in 3d, the order of rotations matters!

An example of a discrete subgroup of the group of 3d rotations is to restrict ourselves to the subgroup of rotations with angle $\pi = 90^\circ$. This is the group of rotational symmetries of the cube, and has 24 elements, and is isomorphic to S_4 (can you see why?). This subgroup is also not abelian: for example, compare the result of successive 90° rotations about the $\hat{z}$ and $\hat{x}$ axes, versus the opposite ordering – the result is evidently not the same.



2.6 Matrix groups

The cases we just discussed — of rotations in 2d (either in terms of the complex phases $e^{i\theta}$ of the 2×2 real matrices $R(\theta)$) and rotations in 3d (via the 3×3 real matricec R) — are examples of **matrix groups**. A matrix group is a set of square invertible matrices, with composition given by matrix multiplication, that might satisfy additional constraints.

In particular, the realization of the rotation group as complex phases is the group $U(1)$, which is a special case of the matrix group $U(n)$ of unitary $n\times n$ matrices with complex components – *i.e.*, the set of $n\times n$ matrices that satisfy $U^\dagger U = UU^\dagger = I$. In the case of $n=1$, "1×1 unitary matrices" are just complex phases $e^{i\phi}$. On the other hand, the realization in terms of 2×2 real matrices is a special case of the matrix group $SO(n)$, consisting of $n\times n$ so-called "special orthogonal" matrices (to be discussed below). The map we constructed between these two groups in (2.9) shows that these two groups are isomorphic to each other,

$$U(1)\cong SO(2)\,.$$

More generally, we can consider the set of $n\times n$ invertible matrices M, under matrix multiplication. This set of matrices form a group, which we can verify by checking the four group properties:

1. **Closure:** If we multiply two $n\times n$ matrices, the output is another $n\times n$ matrix: $M_1\cdot M_2 = M_3$. If M_1 and M_2 are both invertible matrices, then M_3 is necessarily invertible; multiplying both sides by $M_2^{-1}M_1^{-1}$,

$$\begin{aligned} M_1M_2 = M_3 \quad\Rightarrow\quad M_2^{-1}M_1^{-1}M_1M_2 &= M_2^{-1}M_1^{-1}M_3 \\ 1 &= M_2^{-1}M_1^{-1}M_3 \\ \Rightarrow M_3^{-1} &= M_2^{-1}M_1^{-1} = (M_1M_2)^{-1} \end{aligned}$$

   Since M_1^{-1} and M_2^{-1} exist, M_3^{-1} therefore also exists.

2. **Associativity:** Matrix multiplication is associative: $M_1(M_2M_3) = (M_1M_2)M_3$.

3. **Identity:** This is just the $n \times n$ identity matrix,

$$e = \mathbb{1} = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

4. **Inverse:** We have restricted the set to strictly invertible matrices, $M^{-1}M = MM^{-1} = \mathbb{1}$. Non-invertible matrices would *not* form a group.

This group of invertible $n \times n$ matrices is called the **general linear group** $GL(n, F)$, where F denotes the type of matrix entries that are allowed; for example, we could have $F = \mathbb{R}$ (restricting the matrix entries to real numbers), or $\mathbb{C}$ (allowing the matrix entries to be complex numbers). Some special subgroups of $GL(n, F)$ that show up frequently in physics are as follows:

- The **orthogonal group** $O(N)$ is the subgroup of $GL(N, \mathbb{R})$ which imposes the following additional condition on the matrices M:

  $$M \text{ is orthogonal:} \qquad M^T M = \mathbb{1} \quad \leftrightarrow \quad M \in O(N)$$

  This group is non-abelian.

- The **special orthogonal group** $SO(N)$ arises from restricting to orthogonal matrices that have the following additional ("special") property:

  $$M \text{ (is orthogonal and) has unit determinant:} \quad \det(M) = 1 \quad \leftrightarrow \quad M \in SO(N)$$

  You can verify that if $\det(M_1) = 1$ and $\det(M_2) = 1$, then $\det(M_3 = M_1 M_2) = \det(M_1)\det(M_2) = 1$, so this property continues to under matrix multiplication. Here, we assume that $N > 1$.

  For instance, the 2d rotation matrices R_θ in (2.8) parameterize the set of all possible 2×2 special orthogonal matrices, satisfying $R_\theta^T R_\theta = \mathbb{1}$ and $\det R_\theta = 1$, so that these matrices indeed parameterize the matrix group $SO(2)$. $SO(2)$ is an abelian group.

  Meanwhile, the 3d rotation matrices R as defined in (2.10) parameterize the set of all possible 3×3 matrices that are special orthogonal (satisfy $R^T R = \mathbb{1}$ and have unit determinant $\det R = I$). Then, the matrices R are indeed a representation of the elements of $SO(3)$. $SO(N)$ for $N \geq 3$ is a non-abelian group.

Exercise 2.10

Verify that the matrix R in (2.10) is a special orthogonal matrix. In particular, verify that:

(a) $R^T R = \mathbb{1}$;

(b) $\det R = 1$.

(c) Furthermore, show that the length of the vector $\vec{v} = (x, y, z)$ is invariant under rotations by R (*i.e.* $\vec{v} \to R\vec{v}$). Which of the properties (a) and/or (b) imply this invariance?

- The **unitary group** $U(N)$ is the set of $N \times N$ invertible square matrices with complex entries, that additionally satisfies the *unitarity* condition:

$$M \text{ is unitary:} \quad M^\dagger M = M M^\dagger = \mathbb{1} \quad \leftrightarrow \quad M \in U(N)$$

  where $M^\dagger = (M^*)^T$ denotes the complex transpose, or in other words *hermitian conjugate* of M.

  The special case of $N = 1$ corresponds to the group $U(1)$, which we saw is simply the set of complex phases ("1×1" complex matrices). $U(1)$ is an abelian group isomorphic to $SO(2)$, describing the group of rotations in 2d. However, for general $U(N > 2)$ this is a non-abelian group.

- While we're covering the most important matrix groups that appear in physics, we would be remiss to not mention the **special unitary group** $SU(N)$, which is the subgroup of the unitary group that has the additional (you guessed it) "special" condition of unit determinant:

$$M \text{ (is unitary and) has unit determinant:} \quad \det(M) = 1 \quad \leftrightarrow \quad M \in SU(N)$$

  Here we assume that $N > 1$ (there is no $SU(1)$). Later in our discussion of quantum mechanics especially we will return to applications of $SU(2)$ in physics.

In more detail: $SU(2)$**:** For example, a general $SU(2)$ matrix can be written in terms of two complex parameters a, b satisfying the constraint $|a|^2 + |b|^2 = 1$, as:

$$M = \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} \tag{2.11}$$

It is straightforward to show that the inverse of M is its Hermitian conjugate, so that,

$$M^\dagger M = \begin{pmatrix} a^* & b^* \\ -b & a \end{pmatrix} \begin{pmatrix} a & -b^* \\ b & a^* \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & 0 \\ 0 & |a|^2 + |b|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and that $\det M = |a|^2 + |b|^2 = 1$. Since this group can be parameterized by two complex parameters = four real parameters satisfying one constraint, it actually depends on only three independent real parameters. The *dimension* of this group is then 3; the same dimension as $SO(3)$ (which depended on the 3 real parameters $\theta_1, \theta_2, \theta_3$.[6] These 2×2 $SU(2)$ matrices act on two-component complex vectors, which in physics are called *spinors*. These spinors are the analogue of vectors in the real rotation group, and have their name because the types of particles that are represented by spinors have (you guessed it) spin! (Particles like electrons with spin-1/2 are acted on in this way.) We may return to this point later for a discussion of spin angular momentum in quantum mechanics.

——— *End Lecture 6.*

Application: to crystal lattices There are 32 unique so-called *crystal lattices*, repeating lattices in 3-dimensions. Schematically, the first step of this classification arises from finding all the finite subgroups of $O(3)$, the group of orthogonal 3×3 matrices. This is the pertinent group since the group of lattice symmetries that fix a specific lattice point can be thought of as real 3×3 matrices, which must be orthogonal so that the angles between the various lattice vectors are preserved. Applying some additional constraints, in particular that the transformation must map the lattice back to itself, leads to the complete list of 32 symmetry groups.

[6] If you've seen the Pauli spin matrices before, the three generators of this group can be taken to be the three Pauli matrices.

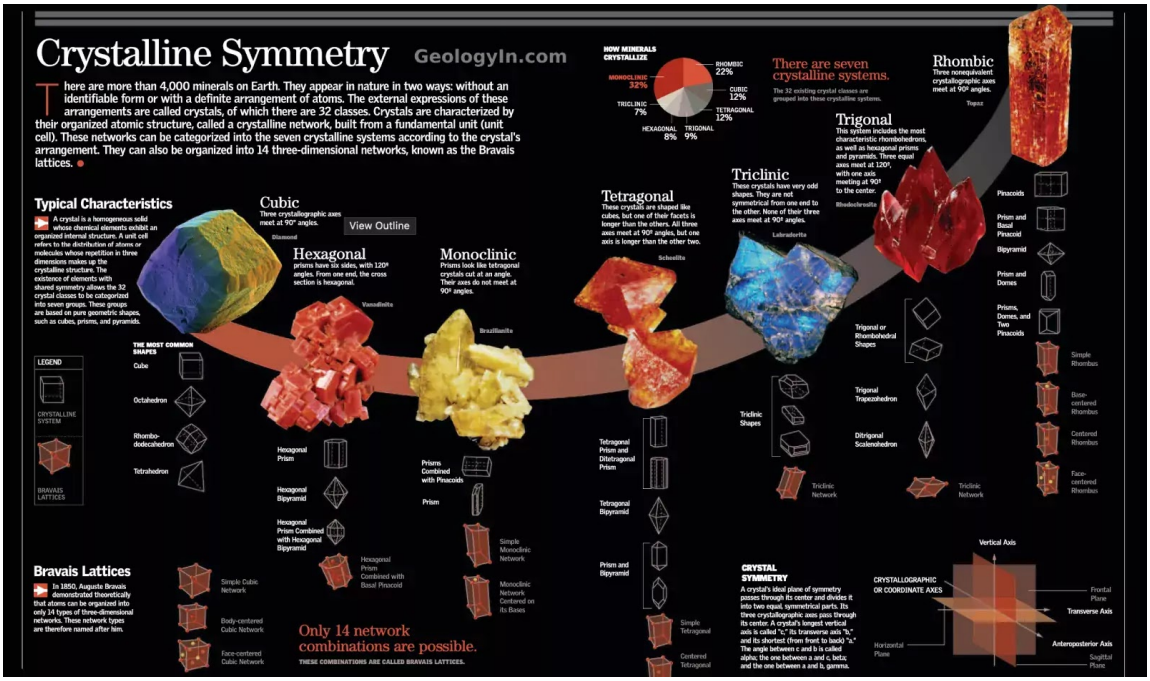


Figure 4: Picturing the 32 possible crystal lattices. Image by GeologyIn.com.

Application: Quantum computing In quantum computing, quantum logic gates are the building blocks of quantum circuits. Quantum gates that act on n qubits are described as $U(2^n)$ matrices, which are unitary because they need to enact unitary operations on the qubits. They act on 2^n dimensional basis vectors, which correspond to all the possible outcomes once the state of the qubits is measured.

For example, single-qubit quantum gates are 2×2 unitary matrices. Some of the most important single-qubit operations are enacted by the Pauli matrices, called the "Pauli gates":

$$\sigma_x = \text{NOT} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

along with the identity.

Application: Lie groups An infinite, continuous group which additionally posseses the property that functions of the elements of the group are smooth, differentiable, and analytic, is called a *Lie group*. Another way to say this is that a Lie group is a continuous group which is also a smooth *manifold*, where a manifold is a space that locally looks like flat Euclidean space.

For example: consider $SO(2) \cong U(1)$. The group is parameterized by a single continuous, real parameter θ. (We call the *dimension* of a Lie group the number of continuous parameters that is needed to parameterize it, so $SO(2) \cong U(1)$ are Lie groups of dimension 1. The dimension is equal to the number of generators of the Lie group.) Given some representation of our group (for example, the matrix R_θ which is a function of θ), we have that

$$R_{\theta_1} \cdot R_{\theta_2} = R_{\theta_3}$$

in other words, the group closes; two rotations composed yield a third rotation. Now, clearly θ_3 is a continuous, analytic function of θ_1 and θ_2 – in this simple case, $\theta_3 = \theta_1 + \theta_2 \mod 2\pi$. The crucial step to recognizing a *Lie* group is to notice that the group action can be viewed as

actually mapping these continuous variables θ to each other; in this case, as

$$f(\theta_1, \theta_2) = \theta_3 = \theta_1 + \theta_2 \bmod 2\pi$$

The fact that this function is nice and differentiable means that we can do things like Taylor expand our group elements around $\theta = 0$, and so on.

More generally, a Lie group of n dimensions is parameterized by some continuous, n-dimensional space M, so that the group action maps points $a, b \in M$ to a third point $c \in M$ so that the mapping function is analytic:

$$g(a) \cdot g(b) = g(c) \quad \Rightarrow \quad f(a, b) = c\,, \quad \text{for } f \text{ a continuous, analytic function of } a, b$$

The space M is locally flat, which just means that if I zoom in to very tiny distances it looks like flat n-dimensional space $\mathbb{R}^n$.

All of the examples of matrix groups we have mentioned above are Lie groups. In the 1-dimensional example of $SO(2) \cong U(1)$, the manifold in question is simply the circle, since it's parameterized by the variable θ which is a coordinate on the circle. In the example of $SO(3)$, the manifold is 3-dimensional, and one basis for the generators is in terms of the $R_x(\alpha), R_y(\beta), R_z(\gamma)$ we gave above. In this case the manifold corresponds to the solid ball in 3d space that identifies antipodal points on the surface of the ball (basically, since rotations by π and $-\pi$ are the same).

It seems that Lie groups must be enormously complicated. However because of the additional data of a manifold structure, it is nevertheless possible to study them in detail, and even classify them.

A bit of history: Lie groups had humble beginnings in Norwegian Sophus Lie's efforts to solve differential equations. He was inspired by the attempts in the late 18th century and early 19th century to solve algebraic polynomial equations. Galois showed that a general 5th degree equation (like, $x^5 - x - \frac{1}{3} = 0$) – unlike equations of lower degrees – cannot be solved with radicals, by translating the solvability of this equation to the solvability of a finite permutation group of the roots called the Galois group. Sophus wondered if these sorts of symmetry methods that were useful for solving algebraic equations be used to solve differential equations. Pursuing this approach led to the development of Lie theory.

Exercise 2.11

(a) A general element of $U(1)$ is a complex number $z = e^{i\theta}$. Writing $z = x + iy$ for real variables x, y, verify that x, y define a circle of radius 1 in the plane ($\mathbb{R}^2$). This is why $U(1)$ as a manifold is a circle (the "1-sphere" S^1).

(b) A general element of $SU(2)$ is a matrix M of the form (2.11). Writing $a = x + iy$ and $b = z + iw$ for real variables w, x, y, z, verify that w, x, y, z define a 3-dimensional sphere of radius 1 in 4-dimensional flat space ($\mathbb{R}^4$). Thus $SU(2)$ as a manifold is the 3-sphere S^3.

——— *End Lecture 7.*

2.7 Appendix: Summary of terms and examples

Abelian group: A group whose elements all commute with one another, $a \cdot b = b \cdot a$ for all $a, b \in G$. (A symmetry is either abelian or non-abelian.)

Associativity: the property that a composition law satisfies $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Cayley table: The table describing the structure of a finite group, by arranging all possible products of a group's elements.

Closure: The property that composition of any members of a set always maps to another member of the set.

Continuous symmetry: A symmetry that describes continuous transformations of a system. (A symmetry is either continuous or discrete.)

Discrete symmetry: A symmetry that describes discrete, non-continuous transformations of a system. (A symmetry is either continuous or discrete.)

Finite group: A group containing a finite number of elements. (A symmetry is either finite or infinite.) Note: all finite groups are discrete, but not all discrete groups are finite.

Generators: The minimal subset of a group such that every group element can be expressed as ("generated by") a composition of finitely many elements of the subset and their inverses.

Group: A set of elements equipped with a multiplication/composition law that satisfies the properties of closure, associativity, identity, and inverse.

Identity: An element e that leaves unchanged every element of a set when the operation is applied: $e \cdot a = a \cdot e = a$.

Infinite group: A group containing an infinite number of elements. (A symmetry is either finite or infinite.)

Inverse: The inverse of an element a is the element a^{-1} such that $a^{-1} \cdot a = a \cdot a^{-1} = e$.

Isomorphism: Two group are said to be isomorphic ($G \cong G'$) if there exists a bijective map between their group elements that preserves the group multiplication law.

Lie group: A continuous group that is also a smooth manifold.

Matrix group: A group whose elements consist of invertible matrices, where the group law corresponds to matrix multiplication.

Non-abelian group: A group that contains elements that do not commute with one another, $a \cdot b \neq b \cdot a$ for some $a, b \in G$. (A symmetry is either abelian or non-abelian.)

Order (of a group): The number of group elements of a finite group.

Representation: A group representation is a realization of the group's elements as linear operators acting on a vector space. Denoting the linear operators as $D(a)$ for $a \in G$, they must (1) satisfy the group multiplication law, $D(a)D(b) = D(ab)$, and (2) realize the identity element as the identity operator on the vector space, $D(e) = \mathbb{1}$.

Subgroup: A subset H of a group G that itself satisfies the conditions to be a group.

Examples of groups considered in this section:

- **Dihedral group D_n:** the group of symmetries of a regular polygon with n sides (where generally we take $n \geq 3$). This group is of order $2n$, consisting of n rotational symmetries and n reflectional symmetries. It is generated by 2 elements. *Descriptors: finite, discrete, non-abelian.*

  For example, D_3 is the order-6 symmetry group of the equilateral triangle, and D_4 the order-8 symmetry group of the square.

- **Cyclic group Z_n:** an order n group generated by a single element a, whose n elements can be expressed as $\{a, a^2, \ldots, a^n = e\}$. *Descriptors: finite, discrete, abelian.* $Z_n \cong \mathbb{Z}_n$.

  For example, Z_2 is the order 2 group consisting of elements $\{a, e\}$, such that $a^2 = e$. Z_2 is the unique finite group of order 2.

- **Integers $\mathbb{Z}$ under addition:** the infinite group consisting of all integers under the composition law of addition. *Descriptors: infinite, discrete, abelian.*

- **Integers modulo n under addition $\mathbb{Z}_n$:** the equivalence classes of integers modulo n, so that $a \cdot b = (a + b) \bmod n$. An order n group isomorphic to the cyclic group, $\mathbb{Z}_n \cong Z_n$. *Descriptors: infinite, discrete, abelian.*

- **Symmetric group S_n:** the set of all permutations of n objects. This group is of order $n!$, and is generated by $n - 1$ elements. *Descriptors: finite, discrete, non-abelian for $n \geq 3$.*

  For example, the symmetric group S_3 is isomorphic to D_3, $S_3 \cong D_3$.

- **General linear group $GL(N, F)$:** the group of invertible $N \times N$ matrices over the field F (for instance, we could take $F = \mathbb{R}$ or $\mathbb{C}$). *Descriptors: infinite, continuous, non-abelian.*

- **Orthogonal group $O(N)$:** the group of $N \times N$ invertible matrices with real components that are also *orthogonal*, satisfying $M^T M = \mathbb{1}$. *Descriptors: infinite, continuous, non-abelian.*

- **Special orthogonal group $SO(N)$:** the group of $N \times N$ invertible matrices with real components that are also (1) orthogonal, satisfying $M^T M = \mathbb{1}$, and (2) have unit determinant, $\det(M) = 1$. *Descriptors: infinite, continuous, non-abelian for $N \geq 3$, abelian for $N = 2$.*

  This group corresponds to the group of rotations in N-dimensional Euclidean space. For example, $SO(2)$ is the group of rotations in the plane, and $SO(3)$ is the group of rotations in 3d.

- **Unitary group $U(N)$:** the group of $N \times N$ invertible matrices with complex components that are also *unitary*, satisfying $M^\dagger M = M M^\dagger = \mathbb{1}$. *Descriptors: infinite, continuous, non-abelian for $N \geq 2$, abelian for $N = 1$.*

  The case of $N = 1$ corresponds to the set of complex phases $e^{i\phi}$ for $0 \leq \phi < 2\pi$, describing the group of continuous rotations ϕ about an axis. $U(1)$ is isomorphic to $SO(2)$, $U(1) \cong SO(2)$.

- **Special unitary group $U(N)$:** the group of $N \times N$ invertible matrices with complex components that are also (1) unitary, satisfying $M^\dagger M = M M^\dagger = \mathbb{1}$, and (2) have unit determinant, $\det(M) = 1$. *Descriptors: infinite, continuous, non-abelian.*

Some useful facts about matrices:

A matrix is an array of numbers. The *dimension* of an $n \times n$ square matrix is the number of its rows or columns, n. A column matrix is an $n \times 1$ matrix of n rows and 1 column, while a row matrix is a $1 \times n$ matrix of 1 row and n columns.

The matrix *transpose* is the exchange of its rows and columns, which for a square matrix yields its reflection about the diagonal; for instance,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}^T = \begin{pmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \qquad \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}^T = \begin{pmatrix} a_1 & a_2 & a_3 \end{pmatrix} \tag{2.12}$$

The *Hermitian conjugate* of a matrix is its conjugate transpose, given by first complex conjugating its entries and then taking the transpose, $A^\dagger = (A^*)^T$. A *symmetric* square matrix is invariant under reflection about the diagonal, $A^T = A$. An *anti-symmetric* square matrix is equal to its negative transpose, $A^T = -A$.

It is often useful to use an index notation to describe the entries of a matrix. By the notation A_{ij}, we mean the entry of the matrix A in the i'th row and the j'th column; so, the first index is a row index and the second is a column index (see (2.12) for an example). For an $m \times n$ matrix, the row index i runs over $\{1, \ldots, m\}$ and the column index j runs over $\{1, \ldots, n\}$.

For example, in index notation the formula for the entries of the transpose of a matrix A is $(A^T)_{ij} = A_{ji}$. In index notation, the multiplication of an $m \times n$ matrix A and $n \times m$ matrix B can be written as,

$$(AB)_{ij} = \sum_{j=1}^{n} A_{ij} B_{jk} \,, \qquad i, j = 1, \ldots, m$$

which yields an $m \times m$ matrix. Matrix multiplication is associative, $(AB)C = A(BC)$, but not necessarily commutative, $AB \neq BA$.

An $n \times n$ matrix A is *invertible* if there exists another matrix A^{-1} such that $AA^{-1} = A^{-1}A = \mathbb{1}$ for $\mathbb{1}$ the $n \times n$ identity matrix of 1's on the diagonal and 0's elsewhere.

The *trace* of an $n \times n$ square matrix is the sum of its diagonal entries, $\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}$. The trace satisfies

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA) \,, \qquad \mathrm{Tr}(A) = \mathrm{Tr}(A^T) \,.$$

The *determinant* of an $n \times n$ square matrix is a number which encodes various properties of the matrix. For example, the determinant of a 2×2 matrix is given by,

$$\det \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = a_{11}a_{22} - a_{12}a_{21} \,.$$

The determinant satisfies $\det(AB) = (\det A)(\det B)$.

3 Symmetries of Spacetime and Special Relativity

Possibly helpful resources:

- **MOST RECOMMENDED:** *A First Course on Symmetry, Special Relativity, and Quantum Mechanics* by Kunstatter and Das, Section 4.5 and Chapter 6 Sections 6.1 – 6.4, 6.8 – 6.11.
- University of Cambridge Professor David Tong's lecture notes on Special Relativity are a nice additional resource, especially Section 7.3. Available at this link.
- **For review:** If you need to brush up on some of the concepts in special relativity that you learned in PHYS 200 (we will do some small amount of review in class), I posted my PHYS 200 lecture notes from Fall 2024 to Moodle. Otherwise, you might also wish to reference Section 5 and the rest of Section 6 from Kunstatter-Das.

3.1 Isometries of Euclidean space and classical physics

Consider 3d (Euclidean) space, with coordinates $\vec{x} = (x, y, z)$. An observer sees a stationary ball on the ground, announcing that at a particular time (say, $t = 0$) the ball is at location $\vec{x}_{\text{ball}} = (x_b, y_b, z_b)$. Meanwhile, a second observer shares the same origin as the first observer, but has rotated her axes to use coordinates $\vec{x}' = (x', y', z')$, where $\vec{x}' = R\vec{x}$ for some rotation matrix R (in general, specified by a matrix (2.10)). She too sees the ball, and declares that it sits at coordinates $\vec{x}'_{\text{ball}} = (x'_b, y'_b, z'_b)$.

There is no reason that the coordinates of the two observers need to agree with each other; there is nothing wrong with using different rotated coordinate systems. What *has* to be true so that the primed and unprimed coordinate systems agree on the same physics?

The answer is that the transformation should preserve distances between pairs of points. Distance-preserving transformations are known as **isometries**. In our example, the two observers should certainly agree that the ball is a certain distance away from their shared origin. That distance is calculated by the 3d Pythagorean theorem,

$$d^2 = x_b^2 + y_b^2 + z_b^2 = x_b'^2 + y_b'^2 + z_b'^2$$

In other words, however we decide to describe our coordinate system, the Pythagorean theorem should hold true!

What is the condition on the 3d rotation matrix R for this to be true? In a given frame (say the unprimed frame), the distance between two points $\vec{x}_1$ and $\vec{x}_2$ is calculated by computing the displacement vector $\Delta\vec{x} = \vec{x}_2 - \vec{x}_1$, and then computing the inner product,

$$d^2 = (\Delta\vec{x})^T(\Delta\vec{x}) = \begin{pmatrix} \Delta x & \Delta y & \Delta z \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{3.1}$$

where $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, $\Delta z = z_2 - z_1$. (Of course, in our example of the ball $\vec{x}_1 = (0, 0, 0)$ is just the origin, so $\Delta x = x_b$, *etc.*) In the primed frame, however, coordinates are transformed by a rotation matrix R, as $\Delta\vec{x}' = R\Delta\vec{x}$ so we would compute

$$d'^2 = (\Delta\vec{x}')^T(\Delta\vec{x}') = (R\Delta\vec{x})^T(R\Delta\vec{x}) = (\Delta\vec{x})^T R^T R\,\Delta\vec{x}$$

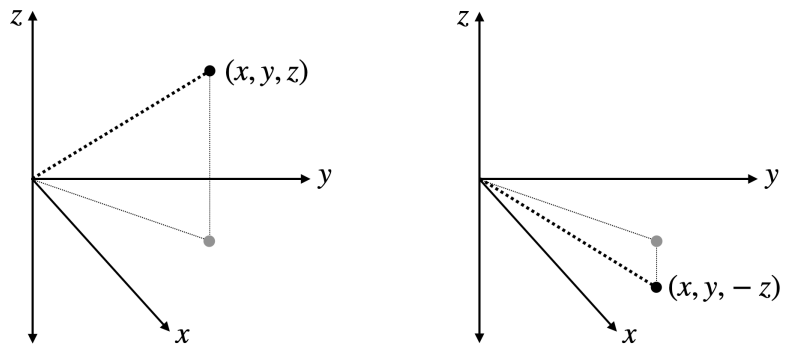


Figure 5: The transformation O in (3.2) enacts a reflection about the x-y plane.

(We used $(AB)^T = B^T A^T$ for two matrices A, B.) We see that we can only have $d^2 = d'^2$ if $R^T R = \mathbb{1}$. In other words: **distance measurements in 3d (flat, Euclidean) space are invariant under rotations of the coordinate axes if the rotation matrix is orthogonal,**

$$R^T R = \mathbb{1}$$

This property is manifestly true of the general 3d rotation matrix R given in (2.10), as you verified in Exercise 2.10. Actually, we see that this property is true of any orthogonal matrix O satisfying $O^T O = \mathbb{1}$; not just the special orthogonal matrices R restricted to $\det R = 1$. We conclude that the group $O(3)$ is part of the isometry group of 3d Euclidean space — it is the group of distance preserving transformations of Euclidean space that preserve a fixed point (the origin).

Note that the group $O(3)$ includes the rotation group $SO(3)$ as a subgroup, but also matrices that don't have unit determinant. For example, consider the matrix

$$O = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{3.2}$$

This matrix O satisfies $O^T O = \mathbb{1}$, but has determinant -1 rather than $+1$, so O is an element of $O(3)$ but not of $SO(3)$. What does this transformation do? Acting on a vector $\vec{x}$,

$$O\vec{x} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

It keeps the x and y coordinates the same, but takes $z \to -z$, thereby corresponding to a reflection about the x-y plane. Similarly, reflections about the other planes are good distance-preserving maps of 3d Euclidean space, and are included in the group $O(3)$.

What other isometries are there of 3d Euclidean space? Besides rotations and reflections about a fixed point, we also should be able to translate the origin by a constant amount. Define a **translation** as an operation that takes all points in the plane and moves them over by a constant vector $\vec{a}$

$$\vec{x} \to \vec{x} + \vec{a}$$

In matrix notation,

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} x + a_x \\ y + a_y \\ z + a_z \end{pmatrix}$$

Exercise 3.1

Verify that the set of translations in three dimensions – *i.e.* the translations of a point by a constant real vector $\vec{a}$ as $\vec{x} \to \vec{x} + \vec{a}$ – form a group.

The group of isometries of 3d Euclidean space is called the Euclidean group, and it consists of the group $O(3)$ which contains rotations and reflections, along with the group of translations. Then, any isometry of 3d Euclidean space has the form

$$\vec{x} \to O\vec{x} + \vec{a} \tag{3.3}$$

for $O \in O(3)$ an orthogonal matrix, and $\vec{a}$ a constant 3-vector. In general n-dimensions, this group is the isometry group of flat n-dimensional space, consisting of $O(N)$ along with constant translations by n-dimensional vectors.

Classical mechanics and the Euclidean group The equations of motion in classical physics are invariant under the set of transformations (3.3), meaning they hold true in any frame of reference related by a rotation, reflection, and/or translation.

For example, consider the motion of a body described by a vector $\vec{x}_1$. A force acts on this body directed from $\vec{x}_2 \to \vec{x}_1$. We might consider a constant force, or a force that is in general a function of the distance $r = |\vec{x}_1 - \vec{x}_2|$ between the bodies, so we will write $\vec{F} = F(r)\hat{r}$ for $\hat{r}$ the unit vector pointed from the body exerting the force to the first body. Newton's second law $F = ma$ states that the motion of the body follows from,

$$F(r)\hat{r} = m\ddot{\vec{x}}_1\,, \qquad \vec{r} = \vec{x}_1 - \vec{x}_2 \tag{3.4}$$

Applying the general transformation $\vec{x} \to \vec{x}' = O\vec{x} + \vec{a}$, in the primed frame the acceleration of the body is transformed by the matrix O, since

$$\vec{x}'_1 = O\vec{x}_1 + \vec{a}\,, \qquad \dot{\vec{x}}'_1 = O\dot{\vec{x}}_1\,, \qquad \ddot{\vec{x}}'_1 = O\ddot{\vec{x}}_1$$

On the left-hand-side, the magnitude of the vector $\vec{r}$ is unchanged, since the translation $\vec{a}$ cancels in the difference of the vectors, and $O^TO = \mathbb{1}$:

$$|\vec{x}'_1 - \vec{x}'_2| = |O(\vec{x}_1 - \vec{x}_2)| = |\vec{x}_1 - \vec{x}_2|$$

(You can see why it was important to consider the transformation of differences between vectors; if $\vec{x}_1$ is supposed to describe the motion relative to the origin in the unprimed frame, and in the primed frame the origin has been translated by a distance $\vec{a}$, we need to take this relative translation into account! Even if $\vec{x}_2 = 0$, this translation of the origin needs to be accounted for.. it is the *distance* between the two bodies that should remain invariant.) Meanwhile, the unit vector $\hat{r}$ transforms as

$$\hat{r}' = \frac{\vec{x}'_1 - \vec{x}'_2}{|\vec{x}'_1 - \vec{x}'_2|} = O\hat{r}$$

Therefore, we have found that,

$$F(r')\hat{r}' = F(r)O\hat{r}\,, \qquad m\ddot{\vec{x}}'_1 = mO\ddot{\vec{x}}_1 \quad \Rightarrow \quad O\left(F(r)\hat{r}\right) = O\left(m\ddot{\vec{x}}_1\right)$$

Both sides of the transformed version of (3.4) are acted on by the same matrix O. Multiplying by the matrix O^T on either side of the equation gets rid of it, as $O^TO = \mathbb{1}$, thus showing that the equations of motion resulting from Newton's second law (3.4) are invariant under the isometries of Euclidean space.

3.2 Computing distances in more general geometries

We just discussed the isometries of flat 3d flat (Euclidean) space, and the fact that classical physics is invariant under these isometries. What if we wanted to study physics in another type of space? This is not a rhetorical question – we will see soon that the consistency of physics with special relativity requires us to consider precisely this question!

The most basic question you can ask that characterizes the properties of a space is *how do we compute distances?* Computing distances in flat n-dimensional space, for instance, corresponds to the n-dimensional version of the Pythagorean theorem,

$$(\Delta s)^2 = (\Delta x_1)^2 + \cdots + (\Delta x_n)^2 \tag{3.5}$$

From here on out we will call the distance squared between two points as $(\Delta s)^2$, to uniformize our notation. Here, $\Delta x_1 = (x_1)_B - (x_1)_A$, $\Delta x_2 = (x_2)_B - (x_2)_A$, and so on for the displacements between two points A and B.

Actually, we should view this distance formula as a *defining property* of the geometry of the space in which we are computing the distance: that it is Euclidean (flat). For instance, in 2d space, this formula is just the usual Pythagorean theorem, which must be true because in the plane the sum of the squares of the lengths of two sides of a right triangle must be equal to the square of the length of the hypotenuse. From this one relationship, we can derive all the other properties of 2d Euclidean geometry that we know and love: that parallel lines never meet, that the sum of the angles in a triangle is 180°, and so on.

It will be useful moving forward to consider the ability to compute infinitesimal distances as well. Letting dx and dy denote infinitesimal increments in x and y, and ds an infinitesimal distance traveled, the 2d distance formula can be written simply as

$$ds^2 = dx^2 + dy^2 \tag{3.6}$$

We call ds the **line element**: the line segment associated with an infinitesimal displacement. The line element $ds^2 = dx^2 + dy^2$ defines the geometry of the space in which we are computing distances, and can be written in whatever coordinates you like. Above we used Cartesian coordinates, but it is equally valid to use radial coordinates; for instance, translating $x = r\cos\theta$ and $y = r\sin\theta$ would give a different presentation of the same line element,

$$ds^2 = dr^2 + r^2 d\theta^2 = dx^2 + dy^2 \tag{3.7}$$

Exercise 3.2

Verify (3.7): Starting from $ds^2 = dx^2 + dy^2$, show that $ds^2 = dr^2 + r^2 d\theta^2$.

This infinitesimal formula is useful to measure lengths of paths in 2d flat space, since we can use it to just sum up the infinitesimal lengths. Say we wish to compute the length of a path L that runs from point P_0 to P_1. We compute,

$$L = \int_{P_0}^{P_1} ds \tag{3.8}$$

As an example, suppose we wish to compute the length of a path described by a function $y(x) = x$ from $x = 0$ to $x = 1$, so from $P_0 = (0, 0)$ to $P_1 = (1, 1)$. (Since this path is just

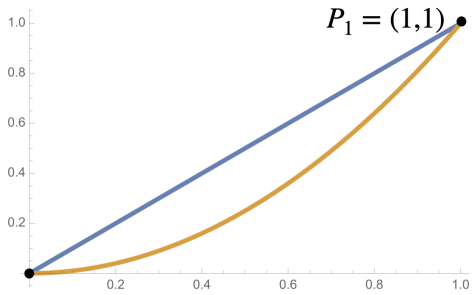


Figure 6: The length of the bottom path is larger than the length of the top path; both are computed by integrating the line element from the origin to P_1.

a straight line, of course this should just compute the distance between the point P_1 and the origin.) We integrate the line element as follows:

$$L = \int_{P_0}^{P_1} ds = \int_{P_0}^{P_1} \sqrt{dx^2 + dy^2} = \int_{P_0}^{P_1} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$= \int_0^1 \sqrt{2}\, dx = \sqrt{2} \approx 1.41$$

where we used that for this path $y = x$, $\frac{dy}{dx} = 1$. Indeed, as we well know the distance between the origin and the point $(1, 1)$ is given by Pythagorean's theorem as $\sqrt{2} = \sqrt{1^2 + 1^2}$; this was just a fancier way of computing that distance.

The fancier formula (3.8) is more useful for nontrivial paths; for instance, what is the distance along the path $y(x) = x^2$ from $x = 0$ to $x = 1$? Now, we still have that $P_0 = (0, 0)$ and $P_1 = (1, 1)$, but $\frac{dy}{dx} = 2x$. The integral is given by,

$$L = \int_{P_0}^{P_1} \sqrt{dx^2 + dy^2} = \int_0^1 \sqrt{1 + (2x)^2}\, dx$$

This is more nontrivial integral whose result is

$$L = \frac{1}{4}\left(2\sqrt{5} + \sinh^{-1}(2)\right) \approx 1.48$$

The length of the second path is longer than the length of the first path, as expected.

What if rather than living in flat 2d space, you lived on a surface of a sphere of constant radius R. How would you measure the shortest distance between two points on the Earth, for example, more generally the length of some specified path between two points on the Earth? On the surface of the sphere, the Pythagorean theorem does *not* hold true, because the surface has *curvature*. For example, lines on the sphere will always intersect eventually, since the space curves. We need the version of (3.6) that holds on the surface of a sphere of radius R.

In this case, you can uniquely fix your position by specifying two angles: θ specifying your latitude, and ϕ specifying your longitude. The corresponding Pythagoras law valid for measuring distances on the sphere tells us that the line element that computes the distance between two points specified by (θ, ϕ) is given by

$$ds^2 = R^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

To compute the distance between two points P_0 and P_1 on the sphere we would need to integrate this line element; for instance, if the path between the two points is described by some function

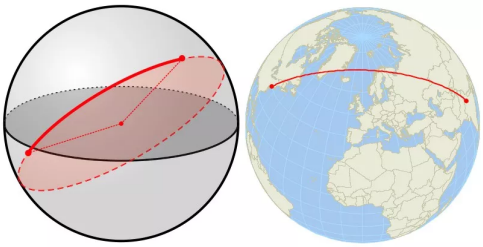


Figure 7: The great circle path is the shortest path between two points (geodesic) on a sphere.

$\phi(\theta)$, we would compute

$$L = \int_{P_0}^{P_1} ds = \int_{P_0}^{P_1} R\sqrt{d\theta^2 + \sin^2\theta\, d\phi^2} = \int_{\theta_0}^{\theta_1} R\sqrt{1 + (\phi')^2 \sin^2\theta}\, d\theta \tag{3.9}$$

where $\phi' = d\phi/d\theta$. More generally, we can use this formula to compute the minimum distance between any two points on the sphere – also called the *geodesic* between two points. In 2d flat space geodesics are straight lines, but in this 2d curved space the geodesics are great circle arcs.

——— *End Lecture 8.*

A geodesic is the path that minimizes the length. The general equation for a geodesic in some space is given by the *Euler-Lagrange equations* (we will see these again when we discuss Lagrangian mechanics!). Suppose we are in n-dimensional space, so ds^2 is a function of n variables $x_1, \ldots, x_n$. The length of a path between two endpoints P_0 and P_1 is computed from $L = \int_{P_0}^{P_1} ds$. Pick one of your variables – say x_1 – to parameterize your path, so pull out a dx_1 from the integrand. The rest of the integrand is in general a function of the other variables, and their derivatives with respect to x_1, $x_2' = dx_2/dx_1$, $x_3' = dx_3/dx_1$, *etc.*:

$$L = \int_{(x_1)_{\text{initial}}}^{(x_1)_{\text{final}}} I(x_1; x_2, x_2', x_3, x_3', \ldots)\, dx_1\,, \qquad I = \frac{ds}{dx_1}$$

What is the path that extremizes the length? This is a classic problem in what's known as the *calculus of variations*, and the answer is given by the Euler-Lagrange equations. The Euler-Lagrange equations tell us that for the path that extremizes this length, the integrand satisfies the following $n-1$ equations:

$$\frac{\partial I}{\partial x_m} = \frac{d}{dx_1}\frac{\partial I}{\partial x_m'}\,, \qquad m = 2, \ldots, n$$

In our sphere example, we only have $n = 2$ variables $x_1 = \theta$ and $x_2 = \phi$, so that the integrand is a function $I(\theta; \phi, \phi' = d\phi/d\theta)$, which satisfies

$$\frac{\partial I}{\partial \phi} = \frac{d}{d\theta}\frac{\partial I}{\partial \phi'}\,, \qquad I = I(\theta; \phi, \phi') = R\sqrt{1 + (\phi')^2 \sin^2\theta}\,. \tag{3.10}$$

Exercise 3.3

In this problem, we want to show that the path of shortest distance between two points on the surface of a sphere of radius R lies along the great circle that connects the points, *i.e.* that the great circle connecting the points is a geodesic. This is a generalization of the definition of a straight line to curved space.

Find the formula that describes a geodesic on the surface of a sphere of radius R by following the following steps:

(a) The integrand I of (3.9) is a function of θ and ϕ', but is actually independent of ϕ. Euler's equation (3.10) thus implies that

$$\frac{\partial I}{\partial \phi} = \frac{d}{d\theta}\frac{\partial I}{\partial \phi'} = 0$$

(does this equation look familiar from PHYS 210?) Use this equation to define a constant of motion, c.

(b) Use your result from (a) to solve for ϕ' as a function of c. Integrate to find $\phi(\theta)$ describing the geodesic.

Hint: To perform the integral, use the change of variables $u = \cot\theta$, and the integral

$$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) .$$

To reiterate the crucial point of this subsection: the line element ds characterizes the difference between flat Euclidean 2d space, and the curved space on the sphere. For example, one manifestation of the fact that the sphere has curvature is the fact that the sum of angles of triangles on the sphere is greater than 180°. Furthermore, parallel lines can in fact meet. The geometry is different!

In different contexts, we might be interested in the isometries of a non-Euclidean geometry. An essential place where this shows up in physics is in Special Relativity. This is what we'll explain next, after some review.

3.3 A brief review of Special Relativity

Galilean relativity

The essence of relativity is the fact that different observers can have different experiences of the same events. An observer standing on a train platform watches a train speed by with constant speed v to the right. Meanwhile, an observer on the train sits comfortably in their seat, and from their perspective appears to watch the person on the platform zoom by with constant speed v to the left. The laws of physics should not depend on which observer is doing the observing, between any inertial (non-accelerating) frame of reference. Put another way, physics should not depend on the physicist.

Applying this principle to classical mechanics results in the **principle of Newtonian relativity** (also called Galilean relativity): The laws of mechanics should take the same mathe-

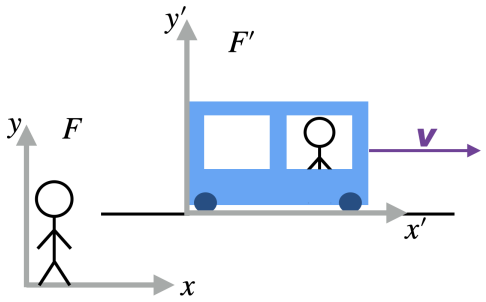


Figure 8: The primed (train) frame F' moves with speed v relative to the unprimed (platform) frame F.

matical form in all inertial frames of reference.

As we have already argued, mechanics should definitely be invariant under the 3d Euclidean group; applying a coordinate transformation (3.3) should not (and does not) change the equations of motion. Besides the isometries of 3d space, Newtonian relativity also needs to allow for transformations between frames moving at constant velocities to one another; if the train observer throws a ball, both the train observer and the platform observer should agree on the same equations of motion for the ball. Such a *Galilean coordinate transformation* can be phrased as

$$\text{Galilean transformation:} \qquad \vec{x}' = \vec{x} - \vec{v}t \tag{3.11}$$

where in this expression the primed frame is the frame at rest with respect to the train observer, which is moving with constant velocity $\vec{v}$ relative to the platform observer.

For example, suppose the train observer throws a ball to the right, and the train moves with constant speed v to the right relative tot he platform. If $\dot{\vec{x}} = \vec{u}\hat{x}$ describes the velocity of a ball according to the platform observer, and $\dot{\vec{x}}' = \vec{u}'\hat{x}'$ the ball's velocity according to the train observer, taking a derivative of the Galilean transformation says that the speeds are related as,

$$u' = u - v \tag{3.12}$$

In other words, the observer on the platform sees the ball moving faster. This makes sense; the platform observer is watching a train move to the right with speed v and a ball on the train also being thrown to the right, so they would see the ball moving faster than the ball's speed in the train's rest frame.

The Galilean transformation is also clearly compatible with Newton's second law; assuming all speeds are constant, taking another derivative says that acceleration is invariant under a Galilean transformation, $\vec{a}' = \vec{a}$. More generally, classical mechanics is invariant under the Galilean group, which includes both Galilean transformations and the Euclidean group.

Special Relativity

There is a problem with Galilean relativity: the speed of light is not invariant under a Galilean transformation, while every experiment that has ever been done concludes that the speed of light is *always* measured to be $c = 2.998 \times 10^8\, m/s$, regardless of the inertial frame in which you do the experiment. Einstein's theory of special relativity grew from requiring that the principle of relativity (that the laws of physics are the same in all inertial frames of reference)

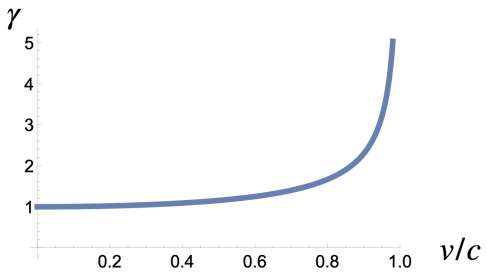


Figure 9: The relativistic gamma-factor $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$ is always larger than 1, approaching infinity as the relative speed v between frames approaches the speed of light.

is compatible with this experimental fact. As you learned in Modern physics, the theory of relativity is based on the two postulates:

Relativistic Postulate 1: The laws of physics are the same in all inertial frames of reference.

Relativistic Postulate 2: The speed of light in vacuum has the same value $c = 3 \times 10^8\, m/s$ in all inertial frames, regardless of the velocity of the observer or the velocity of the source emitting the light.

In your Modern Physics course, you explored the consequences of these postulates for observing physics between different boosted frames, including the effects summarized below. For the purposes of these examples, suppose that primed frame moves with constant speed v relative to the unprimed frame, as in Figure 8.

- **Time dilation:** Moving clocks tick slow. In other words, time moves the fastest in the rest frame of a clock. We call the time elapsed in the rest frame of the clock the *proper time*, typically denoted with the letter τ. Then, the time between ticks in the rest frame of the clock $\Delta\tau$ is related to the time between ticks according to the frame where the clock is moving, Δt, as

  $$\Delta t = \gamma \Delta \tau$$

  γ is the relativistic γ-factor,

  $$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

  Recall that the relativistic gamma-factor $\gamma = 1/\sqrt{1 - v^2/c^2}$ is a dimensionless number always greater than or equal to 1, since $v \leq c$: for very small speeds $v \ll c$ γ approaches 1, while for very large speeds v close to the speed of light c the denominator goes to 0 so that γ blows up to infinity. See Figure 9. Then, we see that for small speeds where $\gamma \approx 1$ the effects of time dilation are not noticeable, but for large speeds closer to the speed of light $\Delta t > \Delta \tau$, which is to say that the time between ticks in the frame in which the clock appears to be moving is larger, so the moving frame sees the clock tick slower.

  For example, perhaps you remember learning about the twin paradox: one twin remains on Earth, while the other twin takes off in a rocket moving with speed $v = \frac{4}{5}c$ to a faraway star many light years away. The astronaut twin reaches the star, immediately

turns around and comes home. Then, the two twins compare how much they have aged; since heartbeats are a kind of clock, we expect that the moving twin will have aged less, since time was moving comparatively faster on Earth (in this problem, the proper time elapsed is the time elapsed in the rest frame of Earth). Since $v = \frac{4}{5}c$ leads to a γ-factor of $\gamma = \frac{1}{\sqrt{1-\frac{16}{25}}} = \frac{5}{3}$, according to the rest frame on Earth

$$\Delta t_{\text{astronaut}} = \gamma \Delta \tau_{\text{Earth}} = \frac{5}{3}\Delta \tau_{\text{Earth}}$$

so the astronaut's heart seems to beat 5/3 more slowly; for every year that passes according to the Earth twin, only 3/5 of a year will have passed for the astronaut twin. By the time the astronaut twin returns to Earth they will have aged less, and literally be younger! This is not just science fiction, but is a well-measured effect int he real world, and important to take into account for GPS systems communicated with fast-moving satellites.

- **Length contraction:** moving rulers contract in the direction of relative motion. (In other words, the length of an object is always largest in its rest frame.) We call the length of an object as measured in the object's own rest frame *proper length*; denoting it L_P, length contraction relates the proper length with the length L as measured in a moving frame,

$$L = \frac{L_P}{\gamma}$$

Since γ is always bigger than 1, $L < L_P$, so the ruler appears shorter in the moving frame.

Exercise 3.4

Suppose you have a meter stick, and can measure lengths up to 1 mm. Determine the speed v that the ruler has to be moving at which relativistic effects become measurable for this ruler.

- **Simultaneity is relative:** Events that are simultaneous in one frame of reference are *not* necessarily experienced as happening at the same time in another frame of reference.

A classic example is the *pole-vaulter paradox*, which can be phrased as follows. There is a barn with two doors that are 4 meters apart. A pole-vaulter practicing on the farm has a pole of length 5 meters. How fast does the pole-vaulter have to run to fit the pole in the barn? Do both the farmer and the running pole-vaulter agree that the pole will actually fit in the barn?

The proper length of the barn is $4m$, and the proper length of the pole is $5m$, so in order to in principle fit the pole in the barn the pole-vaulter has to run fast enough that the pole appears to be length contracted from $5m$ to $4m$:

$$5m/\gamma = 4m \quad \Rightarrow \quad \gamma = \frac{5}{4}\,, \qquad v = \frac{3}{5}c\,.$$

If the pole-vaulter runs at speed $v = (3/5)c$, the farmer at rest with respect to the barn will see the pole as length contracted to the same length as the barn, so it looks like the pole does fit.

However, from the perspective of the runner with the pole, the pole remains its proper length of $5m$, and it is the barn that appears length contracted relative to its proper length,

$$4m/\gamma = 4m/(5/4) = 3.2\,.m$$

So the runner would say that there's no way a $5m$ long pole can possibly fit in a seemingly $3.2m$ long barn. What's going on?

The resolution is that in the frame of the runner, the ends of the pole do not coincide with the walls of the barn at one instant in time: the front of the pole reaches the back of the barn *before* the back of the pole reaches the front of the barn, so there is no one instant in time where the entire pole fits in the barn. Events which are simultaneous to the farmer (that the front of the pole reaches the back of the barn and the back of the pole reaches the front of the barn, so that the whole pole appears to fit in the barn at one instant in time) are *not* simultaneous to the runner. There is no contradiction; the farmer sees the pole as fitting in the barn, the runner does not, and both observers really do see observations that confirm they are in the right.

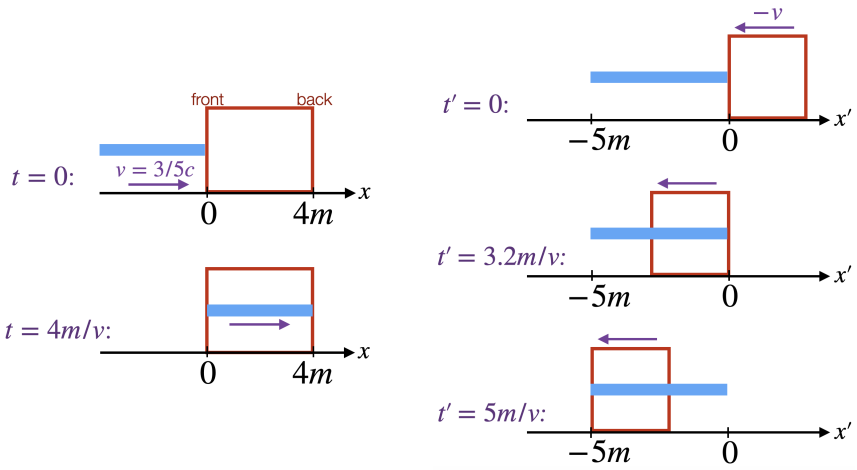


Figure 10: In the rest frame of the barn (left panel), where the pole is length-contracted and moving with speed $+v$, the pole appears to simultaneously fit inside the barn. In the rest frame of the runner (right panel), where the barn is length-contracted and moving with speed $-v$, the pole does not fit inside the barn in one moment of time.

- **Relativistic Doppler shift:** electromagnetic radiation changes frequency depending on how fast the source is moving relative to the observer. This is the effect that explains why galaxies appear red-shifted or blue-shifted depending on whether they are moving away or towards us; you can check, for instance, that in order for a blue light (with wavelength $\lambda = 460\,nm$) to appear red ($\lambda = 650\,nm$), it has to be moving $\frac{1}{3}$ the speed of light away from you.

The Lorentz transformation

All of these effects can be derived from the *Lorentz transformation*: the transformation law between the spacetime coordinates of events as seen in different reference frames, so that such coordinate transformations are consistent with Einstein's postulates. Let us recall the main idea: suppose in a frame F I measure the coordinates of an event as happening at (x, y, z), and at time t. Frame F' moves with speed v in the x-direction relative to frame F. What does an observer in frame F' measure as the coordinates of the same event, in a coordinate system that

is at rest with respect to them? The answer is given by the *Lorentz transformation*,

$$\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2}x\right), \quad x' = \gamma(x - vt) \\ y' &= y\,, \qquad z' = z\,, \qquad \gamma = 1/\sqrt{1 - \frac{v^2}{c^2}} \end{aligned} \tag{3.13}$$

Generally, we will refer to a transformation between inertial frames moving relative to one another as a "boost" between frames.

Figure 11: Two events in the rest frame of the person (frame F).

It's useful to have an example in mind. Suppose at time $t = 0$ a bee flies by at constant speed v, immediately in front of the observer. Let's call the frame at rest with respect to the person F, and the frame at rest with respect to the bee F', moving at speed v in the x-direction relative to frame F. Both the bee and the person have timers, and start their timer as they pass by each other at $t = t' = 0$ (call this Event A). We'll synchronize their coordinate systems, so at $t = t' = 0$ as the bee passes by, they are at $x = y = z = 0$ and $x' = y' = z' = 0$; in other words,

$$(t_A, x_A, y_A, z_A) = (0, 0, 0, 0)\,, \qquad (t'_A, x'_A, y'_A, z'_A) = (0, 0, 0, 0)$$

Some time t later, the observer sees the bee fly past the edge of their yard, a distance L away. This occurs at $t = L/v$, since the bee was flying with speed v. Call Event B the moment the observer sees the bee fly past the edge of the yard, at which point they stop their timer. In the unprimed frame, this event occurs at

$$(t_B, x_B, y_B, z_B) = (L/v, L, 0, 0)$$

According to the bee's rest frame, what are the coordinates of event B? The answer is given by applying the Lorentz transformation:

$$\begin{aligned} (t'_B, x'_B, y'_B, z'_B) &= \left(\gamma(t_B - \frac{v}{c^2}x_B), \gamma(x_B - vt_B), y_B, z_B\right) \\ &= \left(\gamma L(\frac{1}{v} - \frac{v}{c^2}), 0, y_B, z_B\right) = \left(\frac{L}{\gamma v}, 0, 0, 0\right) \end{aligned}$$

where we simplified using $\gamma = 1/\sqrt{1 - v^2/c^2}$. This answer makes sense: in the rest frame of the bee, Event B corresponds to the bee staying still at $x' = 0$, while the person and yard move to the left. But in the bee's frame, the length of the yard is length-contracted from L to L/γ, so the bee will see this event occur at a time $t' = (L/\gamma) \cdot 1/v$ later. In this way, we see that a simple application of the Lorentz transformation is compatible with length contraction.

The Lorentz transformation of coordinates of events also implies the Lorentz transformation of velocities. Suppose the person on the train throw a ball, and you wish to describe the velocity of the ball $\vec{u}$ in either the primed (train) frame or the unprimed (platform) frame, where the train moves with speed v in the x-direction relative to the platform. See Figure 12. By differentiating the Lorentz coordinate transformations, we arrive at the velocity transformation laws,

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}\,, \qquad u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \qquad u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \tag{3.14}$$

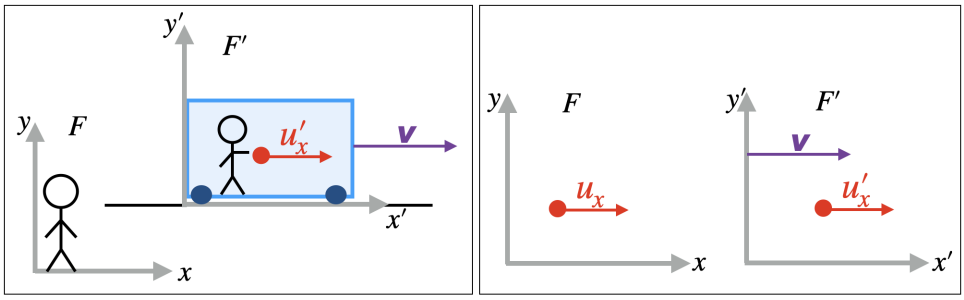


Figure 12: Measuring the speed of an object in the primed versus unprimed frame.

Both the velocities parallel to and perpendicular to the direction of relative motion between the frames transform.

The Lorentz coordinate transformation (3.13) is the analogue of the Galilean transformation (3.11), and the velocity transformation (3.14) is the analogue of the Galilean velocity transformation (3.12), designed to be consistent with Einstein's second postulate. In particular, in the non-relativistic limit that everything travels slowly so $v \ll c$ and $\gamma \to 1$, the relativistic expressions exactly reproduce the classical ones: for the coordinate transformations,

$$\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2}x\right) \quad \overset{v\ll c}{\longrightarrow} \quad t\,, \\ x' &= \gamma(x - vt) \quad \overset{v\ll c}{\longrightarrow} \quad x - vt \end{aligned}$$

and for the velocity transformations,

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \quad \overset{v\ll c}{\longrightarrow} \quad u_x - v$$

with $u'_y \to u_y$ and $u'_z \to u_z$. So, special relativity is consistent with classical relativity in the non-relativistic limit. But in the other limit, it's consistent with the second postulate: applying (3.14) to a light beam, if the unprimed frame sees the beam moving at speed $u_x = c$, so does the primed frame:

$$u'_x = \frac{c - v}{1 - \frac{vc}{c^2}} = c$$

——— *End Lecture 9.*

3.4 The invariant interval and spacetime diagrams

We saw that Newton's laws are invariant under the isometries of 3d Euclidean space (3.3), as well as the Galilean transformations (3.11) and (3.12), so that classical physics is consistent with rotations and translations of your coordinate system, as well as boosts between different inertial frames of reference that obey (3.11). What are the symmetries that are consistent with special relativity, once I impose the condition that the speed of light has to be constant in all frames and so boosts between different inertial frames of reference obey the Lorentz transformations (3.13) and (3.14)?

We certainly believe that special relativity should remain invariant under the 3d Euclidean group (3.3), since regardless of whether objects move close to the speed of light, it still should

not matter whether we rotate or translate our coordinate system. But what is the symmetry associated to invariance under boosts between frames moving at relative speeds? In other words, what is the symmetry principle underlying the Lorentz transformation?

The answer is that there is an analogue of the invariant distance interval (3.5) that has to remain invariant under Lorentz transformations, but crucially this **invariant distance** interval $(\Delta s)^2$ involves both the spatial coordinates *and* time coordinate,

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{3.15}$$

(And yes, there is a funny minus sign in front of $(c\Delta t)^2$!) If two events occur simultaneously at the same value of t so that $\Delta t = 0$, then Δs is simply the distance we would normally measure between two points in space. More generally, Δs should be thought of as the distance in *spacetime* which is invariant under Lorentz transformations.

Some comments:

- Conceptually, it should not be surprising that the time coordinate t is put on the same footing as the spatial coordinate x, y, z in special relativity. Special relativity tells us that time is relative, length is relative, and simultaneity is relative, and that time transforms under Lorentz transformations along with the spatial coordinates. Equation (3.15) tells us that we should think of ct – which has units of distance, since c is given in meters/second and time is measured in seconds – as a coordinate on par with the spatial coordinates x, y, z.
- The invariant interval $(\Delta s)^2$ is the quantity that all observers in all inertial frames must agree upon, which is compatible with the Lorentz transformation. The important point is that without the $-c^2(\Delta t)^2$ bit, observers in different frames can disagree about the usual Euclidean distance between two points, but will always agree about the value of Δs. It provides an observer-independent characterization of the distance between any two events. This point is exemplified in the following exercise.

Exercise 3.5

The positions of two drifting asteroids (x_1, y_1, z_1) and (x_2, y_2, z_2) are tracked by an observer in frame F. The distance between them d **at a given moment in time** is calculated by the usual Pythagorean formula,

$$d^2 = \sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2} \tag{3.16}$$

where $\Delta x = x_2 - x_1$, *etc.* An observer in frame F' moving with speed v in the x-direction relative to the observer in frame F also computes the distance between the asteroids, with result d'.

(a) Suppose that the relative motion between the frames v is much smaller than the speed of light, so that we can use a Galilean coordinate transformation to go between the frames. Show that the distance d' measured in frame F' is the same as the distance d measured in F.

This result shows that the usual Pythagorean measure of distance is invariant under a Galilean coordinate transformation.

(b) Show that the distance d between the two events as defined in (3.16) is *not* invariant under a Lorentz transformation.

In other words, when special relativistic effects become important, observers will *not* agree on the usual Pythagorean distance formula!

(c) Show that if we define the spacetime distance Δs between events as,

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \tag{3.17}$$

that Δs is invariant under the Lorentz transformation. So, while observers disagree on (3.16), they agree on (3.17).

The light cone Let's get a feel for what different values of Δs mean. For simplicity let's do some examples in 1 spatial dimension, *i.e.* in (1+1)-dimensional spacetime, so that we can set $\Delta y = \Delta z = 0$ and just deal with

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 \quad \text{in 1 spatial dimension}$$

Consider the following three examples:

- Consider the path of a light beam, which travels at the speed of light in the x direction between points (ct_1, x_1) and (ct_2, x_2), where since the beam travels at the speed of light,

$$\frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = c \tag{3.18}$$

The spacetime interval Δs between these coordinates computes to

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 = -c^2(\Delta)^2 + c^2(\Delta t)^2 = 0$$

We have learned that $(\Delta s)^2 = 0$ **for anything traveling at the speed of light.**

- Consider a particle moving with constant speed $u < c$ between coordinates (ct_1, x_1) and (ct_2, x_2). The coordinates are related by,

$$\frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} = u \tag{3.19}$$

so we can compute the invariant interval between the coordinates as,

$$(\Delta s)^2 = -c^2(\Delta t)^2 + (\Delta x)^2 = -c^2(\Delta t)^2 + v^2(\Delta t)^2 = c^2(\Delta t)^2 \left(\frac{u^2}{c^2} - 1\right)$$

Since $u < c$, the factor $(u^2/c^2 - 1)$ is always negative. So, we learn that $(\Delta s)^2 < 0$ **for a particle traveling at a speed slower than the speed of light.**

- No particle can move faster than the speed of light, so the previous bullet point implies that we must always have that the invariant interval calculated between two events satisfies,

$$(\Delta s)^2 \leq 0$$

A particle traveling faster than light would have $(\Delta s)^2 > 0$, which is physically not possible.

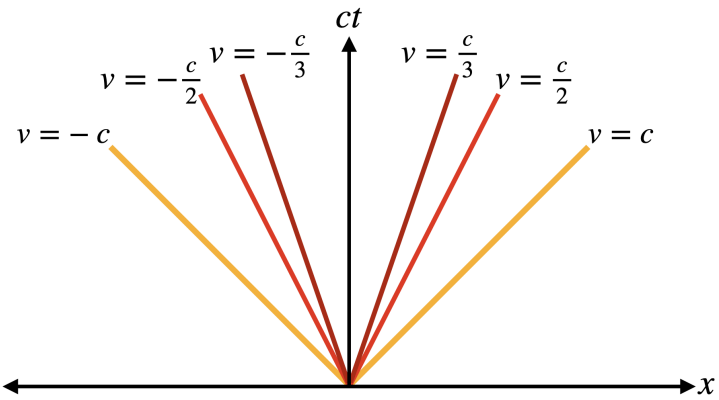


Figure 13: The wordlines of particles moving at speeds v less than c are constrained to lie within the lightcone (depicted as yellow lines) on a spacetime diagram. In this diagram we've assumed the particle starts at $x = 0$ at $t = 0$, so that the lines intersect the origin.

The results of these bullet points can be illustrated with the help of *spacetime diagrams*, which visually represent the paths of particles or relationships between events as a function of both space and time. A spacetime diagram is a plot of ct against the spatial coordinates x, y, z in a fixed inertial frame – although, since it is difficult to draw a 4-dimensional plot, we will focus on spacetime diagrams of just ct plotted against x with the y and z coordinates set to constant values and not depicted. Each point on a spacetime diagram represents an event with coordinates (ct, x) (restricting to 1 spatial dimension). *Note:* this labeling of coordinates with the vertical axis labels ct listed first and horizontal axis labels x listed second is backwards from the usual way we list coordinates on a plot, but is chosen to be consistent with a standard convention we will introduce later in this section.

In our example of the light beam, if we put ct on the vertical axis and x on the horizontal axis, (3.18) implies that the slope satisfies $c\Delta t/\Delta x = 1$. So, the path that a light-beam traces on a spacetime diagram is a straight line with slope 1, which is a straight line at a 45° angle. We call the path of the light beam through spacetime its **worldline**. Similarly, if the light beam were moving to the left with speed c rather than to the right with speed c, its path would satisfy $\Delta x/\Delta t = -c$, so that its worldline corresponds to a line with slope -1 at 45° angle to the negative x-axis. We call the cone bounded by these two worldlines – the region around the ct-axis bounded by the worldlines of a light ray – the **light cone**.[7] $(\Delta s)^2$ computed on the light cone always evaluates to zero.

What we learned in the second and third bullet points above is that the worldline of anything moving with a speed less than c lies within the light cone: since the worldline of a particle moving with speed u to the right satisfies (3.19), its slope on a spacetime diagram is

$$\text{slope:} \qquad c\Delta t/\Delta x = c/u > 1$$

its worldline corresponds to a line at set at an angle of 45° to 90° to the horizontal axis. Similarly, a particle moving with speed u to the left corresponds to a worldline between 45° and 90° to the negative x-axis, so that the particle is bounded by the lightcone. Some examples of worldlines of particles are depicted in Figure 13.

The invariant interval has a peculiar property: it is not positive definite. $(\Delta s)^2$ between events located within the light cone always evaluates to a negative number. This would be

[7] Generalizing from just one spatial coordinate x to 2 spatial coordinates x, y, the lightcone really forms a cone.

strange if we insisted that we deal with Δs rather than $(\Delta s)^2$, since it would imply that Δs is an imaginary number inside the lightcone! However, this is perfectly fine if we just talk about $(\Delta s)^2$ (just go with it..).

Practically speaking, knowing that the spacetime interval Δs is invariant under boosts between frames is extremely useful for solving problems in special relativity, and visualizing events on spacetime diagrams can be a helpful way to understand how events are experienced in different inertial frames. A nice example is illustrated by the following exercise.

Exercise 3.6

A subatomic particle has a lifetime of τ. In its rest frame, it is born at $(ct', x') = (0, 0)$, and decays a proper time τ later, at $(ct', x') = (c\tau, 0)$.

(a) Draw the worldline of the particle in its rest frame on a spacetime diagram.

(b) On a separate diagram, draw the worldline of the particle in a frame in which the particle appears to be moving to the right with speed v.

(c) Compute $(\Delta s)^2$ between the birth and death of the particle in each frame of reference. What can you conclude?

The geometry of spacetime Of course, by defining an invariant interval $(\Delta s)^2$ akin to the invariant distance interval in flat space we are making a statement about the fundamental geometry of spacetime. In the same way that flat, Euclidean space is *defined* by the fact that distances are computed using (3.5), the spacetime of special relativity is *defined* by the fact that "distances" in spacetime are computed using (3.15). In direct analogy with (3.6), we can define the invariant interval between two infinitesimally close points in spacetime by replacing the Δ(somethings) with d(somethings), defining the **line element** as

$$ds^2 = -cdt^2 + dx^2 + dy^2 + dz^2 \tag{3.20}$$

The spacetime defined by the line element (3.20) is called **Minkowski spacetime**, and it is the four-dimensional version of flat Euclidean space $\mathbb{R}^3$ that is compatible with special relativity. We often say that Minkowski spacetime has $3+1$ dimensions, to emphasize the point that there are three space directions and one time direction.[8][9]

——— *End Lecture 10.*

This is a bit of a peculiar space, since we've seen that distinct events can be separated by zero distance $\Delta s = 0$. Events separated by $(\Delta s)^2 = 0$ are said to be *lightlike separated* (or also called *null separated*), since you can reach from one to the other with a light ray. More generally, we call two events with separation $(\Delta s)^2 < 0$ so that one sits within the light cone of the other to be *timelike separated*. Two events that are timelike separated can be causally connected;

[8] Also: while it might not be obvious with the funny minus sign, Minkowski spacetime is flat – it has zero curvature. (Relatedly, the fastest way to get between two points on a spacetime diagram is with a straight line!) General relativity is the subject that deals with more general curved spacetimes, which result from accounting for accelerating frames of reference.

[9] Note: Kunstatter-Das has a nice discussion on p. 107-108 of visualizing events that are separated by the same proper time or proper distance on a spacetime diagram. In Euclidean space, points that are the same distance from the origin lie on a circle, while in Minkowski spacetime, points that are the same proper time or proper distance from the origin lie on hyperbolas.

one can cause the other, since I can send out information at (or less than) the speed of light to reach from one to the other. On the contrary, events with $(\Delta s)^2 > 0$ that therefore sit outside each others' lightcones are said to be *spacelike separated*. Events that are spacelike separated cannot be causally connected – even information traveling at the speed of light cannot be sent between them!

Exercise 3.7

Show that if two events are spacelike separated, there is a Lorentz frame in which they are simultaneous, while if two events are timelike separated, there is *no* Lorentz frame in which they are simultaneous. You may restrict to just 1 spatial dimention, (ct, x).

(This is why events that are spacelike separated cannot be causally connected, while those that are timelike separated can be – two events that can happen simultaneously can't influence one another!)

3.5 Lorentz transformation as symmetries of nature

The same way that classical physics is invariant under the isometries of flat space, special relativity should be invariant under the isometries of Minkowski spacetime. In this subsection, we will describe this isometry group, and understand the 4-vectors that it acts on.

The group of Lorentz transformations Recall that in our discussion of distance in flat 3d space, we saw that requiring

$$d^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

be invariant under a transformation

$$\vec{x} \to \vec{x}' = O\vec{x} + \vec{a}$$

restricted O to an orthogonal matrix (an element of the orthogonal group $O(3)$), and $\vec{a}$ a constant vector (a member of the group of translations of 3d space). The Euclidean group consists of $O(3)$ along with the group of translations, which act on 3-vectors (x, y, z).

We have seen that the interval ds^2 defined in (3.20) is the measure of distance in Minkowski spacetime which is invariant under Lorentz transformations (as you explicitly verified in Exercise 3.5). As we will now describe, these transformations form a group, which act on vectors of both space and time that we will call *four-vectors*.

It is natural that the coordinates of an event in spacetime require four components rather than three: the three components of space, and the component involving time. The coordinates of an event in spacetime are then given by 4-vectors which label these coordinates. We can denote the components of a **4-vector** X as follows:

$$X^\mu = \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}, \qquad \mu = 0, 1, 2, 3 \tag{3.21}$$

We are using an index rotation where Greek indices (like μ) run from $\mu = 0$ to $\mu = 3$; the $1, 2, 3$ components are the usual spatial 3-vector, while the 0-component is the time component ct, normalized so that all components have the same units. X^μ then denotes the four components of the column vector, with $X^0 = ct$, $X^1 = x$, $X^2 = y$, and $X^3 = z$.

How do we compute the invariant distance between two points described by 4-vectors X_a^μ and X_b^μ, with displacement $\Delta X^\mu = X_b^\mu - X_a^\mu$? In terms of the components of the 4-vectors we've already learned to use (3.15),

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

We can express this as an inner product of the 4-vector ΔX^μ as follows:

$$(\Delta s)^2 = (\Delta X)^T \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Delta X = \begin{pmatrix} c\Delta t & \Delta x & \Delta y & \Delta z \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$
$$= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \quad \checkmark$$

This formula should be compared with (3.1); in flat Euclidean space, the inner product of three-vectors was just computed with $(\Delta s)^2 = \vec{x}^T\vec{x}$, whereas in 4d Minkowski space, we have an extra matrix insertion in the middle. In other words, we can interpret the invariant distance in special relativity as an inner product of 4-vectors, *as long as* we also include the matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

in the definition of the inner product, ensuring that the 0-th components of the 4-vectors acquire the extra minus sign. The invariant distance is computed as the inner product[10]

$$\text{Minkowski:} \qquad (\Delta s)^2 = \Delta X \cdot \Delta X \equiv (\Delta X)^T \eta \Delta X = \sum_{\mu,\nu=0}^{3} (\Delta X^\mu)^T \eta_{\mu\nu} \Delta X^\nu$$

In the last equality we are using an index notation: an object like X^μ with a single index μ is a 4-vector with four components running over $\mu = 0, 1, 2, 3$, while an object with two indices like $\eta_{\mu\nu}$ is a 4×4 matrix with rows labeled by $\mu = 0, 1, 2, 3$ and columns labeled by $\nu = 0, 1, 2, 3$, so that when we do the sums over repeated indices we are explicitly writing out the matrix multiplication. Explicitly: $\eta_{00} = -1$, while $\eta_{11} = \eta_{22} = \eta_{33} = 1$, and every off-diagonal component of η is 0 ($\eta_{01} = \eta_{10} = \eta_{12} = \cdots = 0$), so that writing out the matrix multiplication in index notation yields:

$$\sum_{\mu,\nu=0}^{3} (X^\mu)^T \eta_{\mu\nu} X^\nu = (X^0)^T \eta_{00} X^0 + (X^1)^T \eta_{11} X^1 + (X^2)^T \eta_{22} X^2 + (X^3)^T \eta_{33} X^3$$
$$= -(ct)^2 + x^2 + y^2 + z^2$$

More generally, computing $(\Delta s)^2 = X \cdot X$ computes the invariant distance between the point X in spacetime and the origin.

The 4×4 matrix η (the Greek letter eta) is called the **Minkowski metric**. This metric defines the geometry of spacetime: it tells us the right way to compute the inner products of 4-vectors so that we can compute distances $(\Delta s)^2$ or ds^2 that are invariant under Lorentz transformations. By contrast, the metric of Euclidean space is just the identity matrix,

$$\text{Euclidean:} \qquad (\Delta s)^2 = \Delta\vec{x} \cdot \Delta\vec{x} \equiv (\Delta\vec{x})^T \mathbb{1} \Delta\vec{x} = (\Delta\vec{x})^T \Delta\vec{x}$$

[10] In this entire section, the symbol $\cdot$ will denote *inner product*, not group multiplication. This should not cause confusion in the context of special relativity since all groups we will discuss are matrix groups which act by matrix multiplication, and we don't need an extra symbol for matrix multiplication. Apologies..

so we didn't bother to write it in our discussion of Euclidean space. Note carefully that we are using a notation where the inner product $\cdot$ of 3-vectors just means the usual inner product (*i.e.*, inserting the identity matrix), while the inner product $\cdot$ of 4-vectors requires the insertion of the Minkowski metric η. The general statement is that in some n-dimensional space characterized by some $n \times n$ metric g, the right way to take an inner product in that space requires inserting metric.

Now that we've properly defined the spacetime that Lorentz transformations act on – (3+1)-dimensional Minkowski spacetime, with coordinates parameterized by 4-vectors – we are ready to represent the Lorentz coordinate transformations by matrices that act on these 4-vectors. The Lorentz transformation can be thought of as a 4×4 matrix, conventionally called Λ (the capital Greek letter Lambda), that transforms the spacetime coordinates X^μ in frame F to other coordinates X'^μ in frame F'. Consider, for example, a boost by velocity v in the x-direction. The Lorentz transformation (3.13) can be written with the matrix

$$\Lambda = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.22}$$

so that in matrix notation, $\Lambda X = X'$:

$$\Lambda X = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \gamma\left(ct - \frac{v}{c}x\right) \\ \gamma\left(x - vt\right) \\ y \\ z \end{pmatrix} = X'$$

This is just a fancy way of writing the Lorentz transformation (3.13). An equivalent way to write this expression is in index notation,

$$\sum_{\nu=0}^{3} \Lambda_{\mu\nu} X^\nu = (X')^\mu$$

where $\Lambda_{00} = \gamma$, $\Lambda_{01} = -\gamma\frac{v}{c}$, $\Lambda_{10} = -\gamma\frac{v}{c}$, $\Lambda_{11} = \gamma$, $\Lambda_{22} = \Lambda_{33} = 1$, and all other components of Λ are zero.

The inverse of the Lorentz transformation is simply given by the inverse matrix; in terms of the matrices, we can write

$$\Lambda X = X' \quad \Rightarrow \quad (\Lambda^{-1}\Lambda)X = \Lambda^{-1}X' \quad \Rightarrow \quad X' = \Lambda^{-1}X$$

$$\Lambda^{-1} = \begin{pmatrix} \gamma & \gamma\frac{v}{c} & 0 & 0 \\ \gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.23}$$

This equation is the compact matrix version of the Lorentz transformation from the unprimed frame to the primed frame, which as expected coincides with taking $v \to -v$.

Exercise 3.8

Write out all the terms in the following sums, substituting $(X^0, X^1, X^2, X^3) = (ct, x, y, z)$:

(a) $\sum_{\mu=0}^{3} V_\mu X^\mu$, where V_μ is a collection of four arbitrary numbers.

(b) $\sum_{\nu=0}^{3} M_{\mu\nu} X^\nu$, where $M_{\mu\nu}$ is a collection of 16 arbitrary numbers. Note that since μ

is an unsummed ("free") index, you will need to write what this sum is for each of $\mu = 0, 1, 2, 3$.

Lorentz transformations leave the metric invariant Now we can really answer the question: why are Lorentz transformations given by the particular matrices Λ? In other words, what is the condition for a 4×4 object Λ to be a Lorentz transformation?

The Lorentz transformations are defined to be precisely those 4×4 matrices that leave the inner product invariant:

$$X' \cdot X' = X \cdot X\,, \qquad X' = \Lambda X \tag{3.24}$$

Using the definition of the dot-product in Minkowski space, the left-hand-side of the equation evaluates to,

$$(\Lambda X) \cdot (\Lambda X) = (\Lambda X)^T \eta (\Lambda X) = X^T \Lambda^T \eta \Lambda X$$

while the right-hand-side is equal to $X \cdot X = X^T \eta X$. Therefore, we see that (3.24) is true when Λ obeys the matrix equation,

$$\Lambda^T \eta \Lambda = \eta \tag{3.25}$$

This equation should be taken as the fundamental definition of a Lorentz transformation: Lorentz transformations leave the metric η (in the sense of (3.25)) invariant, which leads to the invariant distance formula.

(3.25) is the condition for a matrix Λ to be a Lorentz transformation. We can verify explicitly that this is true for the matrix (3.22):

$$\begin{aligned}
\Lambda^T \eta \Lambda &= \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -\gamma & +\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -\gamma^2 + \gamma^2(v^2/c^2) & 0 & 0 & 0 \\ 0 & -\gamma^2(v^2/c^2) + \gamma^2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \eta \quad \checkmark
\end{aligned}$$

where we used $\gamma^2 = 1/(1 - v^2/c^2)$ to simplify in the last part. In other words, Lorentz transformations are the transformations of spacetime that preserve the Minkowski metric, in precisely the same way that the orthogonal group $O(3)$ are the transformatiosn of 3d space that preserve the Euclidean (identity) metric. These transformations form a group, the group of all distance-preserving maps (*isometries*) of Minkowski spacetime that leave the origin fixed.

The set of Lorentz transformations satisfying (3.25) forms a matrix group called the **Lorentz group**, which we can show by verifying that matrices defined by the property (3.25) satisfy the four group properties:

- **Closure:** Consider the product of two Lorentz transformations Λ_1 and Λ_2, where each of Λ_1 and Λ_2 obey (3.25), so $\Lambda_1^T \eta \Lambda_1 = \eta$ and same for Λ_2. Then, the product obeys

$$(\Lambda_1\Lambda_2)^T \eta (\Lambda_1\Lambda_2) = \Lambda_2^T \Lambda_1^T \eta \Lambda_1 \Lambda_2 = \Lambda_2^T (\Lambda_1^T \eta \Lambda_1)\Lambda_2 = \Lambda_2^T \eta \Lambda_2 = \eta \quad \checkmark$$

  where we used $(AB)^T = B^T A^T$. So, the product of two Lorentz transformations is also a Lorentz transformation.

- **Associativity:** Matrix multiplication is associative, done.

- **Identity:** The identity of a matrix group is always the identity matrix, in this case the 4×4 matrix with 1's on the diagonal and 0's elsewhere, which satisfies $\Lambda \mathbb{1} = \mathbb{1}\Lambda = \Lambda$ for any matrix Λ.

- **Inverse:** The inverse of an element of a matrix group is the inverse matrix; Λ^{-1} is the inverse matrix to Λ. For instance, in the specific case of Λ being given by (3.22) its inverse is (3.23), which just corresponds to boosting with $-v$ rather than v.

Note that in identifying Lorentz transformations with a matrix group it was absolutely crucial that the transformations were *linear*: they transform coordinates (ct, x, y, z) in one frame to coordinates (ct', x', y', z') in another frame linearly, so that the system of equations only depend on single powers of (ct, x, y, z) (rather than quadratic powers, *etc.*). If this weren't the case, we wouldn't be able to package them into linear matrices.

——— *End Lecture 11.*

Classes of Lorentz transformations The general solutions to (3.25) fall into two classes. The first class is familiar: they are solutions of the form

$$\Lambda = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & O & \\ 0 & & & \end{array}\right)$$

where O is an element of $O(3)$, the orthogonal group of 3×3 matrices satisfying $O^T O = \mathbb{1}$. If you like, you can explicitly check that $\Lambda^T \eta \Lambda = \eta$ for this class of Λ:

$$\left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & O^T & \\ 0 & & & \end{array}\right) \left(\begin{array}{cccc} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & O & \\ 0 & & & \end{array}\right) = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & O^T & \\ 0 & & & \end{array}\right) \left(\begin{array}{c|ccc} -1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & O & \\ 0 & & & \end{array}\right) = \eta$$

where we used $O^T O = \eta$. The other solutions are the Lorentz boosts. The boost by v along the x-axis is given by the matrix (3.22), while the boosts along the other directions y and z are given by rearranging the components of the matrix so that they act in the right way: for a boost by $\vec{v} = v_x \hat{x}$, or $v_y \hat{y}$, or $v_z \hat{z}$, with γ a function of $v^2 = |\vec{v}|^2$, the corresponding transformation matrices are,

$$\Lambda_x = \begin{pmatrix} \gamma & -\gamma\frac{v_x}{c} & 0 & 0 \\ -\gamma\frac{v_x}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_y = \begin{pmatrix} \gamma & 0 & -\gamma\frac{v_y}{c} & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma\frac{v_y}{c} & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \Lambda_z = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\frac{v_z}{c} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\frac{v_z}{c} & 0 & 0 & \gamma \end{pmatrix}$$

A general Lorentz transformation is given by composition of these three basis transformations, keeping in mind that in these definitions γ in the first expression is a function of v_x, γ in the second expression is a function of v_y, and so on.

Together, the Lorentz group is denoted $O(1,3)$ – the comma is meant to differentiate the time component of the matrices with the three spatial components, since the time component has the relative minus sign in the metric. It is composed of conventional rotations (3 parameters for the 3 spatial axes), along with boosts (again, 3 parameters for boosts along the 3 spatial axes). It is a non-abelian group, since general Lorentz transformations will not commute with one another. This group leaves the origin fixed, but if we also include translations of spacetime by a constant 4-vector A^μ, we arrive at the **Poincaré group**: the group of isometries of Minkowski spacetime. A general Poincaré transformation of a 4-vector X^μ may be written

$$X' = \Lambda X + A\,, \qquad \Lambda^T \eta \Lambda = \eta\,, \qquad A = \text{constant 4-vector}$$

and is a 10-parameter group: 3 rotations associated to $O(3)$, 3 boosts associated to the three directions v_x, v_y, v_z, and 4 translations for each direction of spacetime. The Poincaré group is the fundamental symmetry group of nature. It is a non-abelian Lie group.

Rapidity A nice way to understand boosts is to think of them as (hyperbolic) rotations between the space and time directions. To illustrate this, let's just focus on Lorentz transformations in 1 spatial direction, so that we only have to worry about 2×2 rather than 4×4 matrices ($O(1,1)$ rather than $O(1,3)$ – the idea is the same when generalizing to 3 spatial dimensions). A Lorentz transformation by speed v in the x-direction is described by the matrix,

$$\Lambda[v] = \begin{pmatrix} \gamma & -\gamma v/c \\ -\gamma v/c & \gamma \end{pmatrix} \tag{3.26}$$

A clever parameterization of this transformation is to define the **rapidity** φ in terms of v:

$$\gamma = \cosh\varphi \quad \Rightarrow \quad \tanh\varphi = \frac{v}{c}$$

where we used the hyperbolic trig identity $\cosh^2 x - \sinh^2 x = 1$, and $\tanh x = \sinh x/\cosh x$ to express the ratio v/c in terms of hyperbolic tangent. The non-relativistic limit $\gamma \to 1$ ($v/c \to 0$) corresponds to $\varphi \to 0$, while the relativistic limit $\gamma \to \infty$ ($v/c \to 1$) corresponds to $\varphi \to \infty$. In this parameterization, (3.26) takes the form,

$$\Lambda[\varphi] = \begin{pmatrix} \cosh\varphi & -\sinh\varphi \\ -\sinh\varphi & \cosh\varphi \end{pmatrix} \tag{3.27}$$

You can compare this matrix to the standard 2d rotation matrix (2.8) in the x-y plane: it is quite similar, with regular trigonometric functions replaced by hyperbolic trigonometric functions. In the same way that rotation angles add upon composition, $R_{\theta_1} R_{\theta_2} = R_{\theta_1+\theta_2}$, rapidities φ add:

$$\Lambda[\varphi_1]\Lambda[\varphi_2] = \Lambda[\varphi_1 + \varphi_2]$$

Exercise 3.9

Using the matrix (3.27), show that $\Lambda[\varphi_1]\Lambda[\varphi_2] = \Lambda[\varphi_1 + \varphi_2]$, so that rapidities add like angles. It is in this sense that you can think of Lorentz boosts as rotations of space and time.

3.6 Conserved 4-vectors

So far we have concerned ourselves with how Lorentz transformations act on coordinate 4-vectors, X^μ, where the index μ runs over $0, 1, 2, 3$. We defined the coordinate 4-vector $X^\mu = (ct, x, y, z)$ in the way that we did precisely because it transformed nicely under Lorentz transformations, and because the inner product of the 4-vector with itself gave us a Lorentz-invariant quantity, $ds^2 = dX \cdot dX = dX^T \eta dX$.

The world of relativistic physics is much richer, however, than just understanding how coordinates transform, and it turns out that there are other types of 4-vectors that have exactly the same transformation properties under Lorentz transformations! You might recognize a few of them, for instance:

$$\begin{aligned}
\text{spacetime coordinates:} \quad & X^\mu = (ct, x, y, z) \\
\text{spacetime divergence:} \quad & \partial^\mu = \left(-\frac{1}{c}\partial_t, \partial_x, \partial_y, \partial_z\right) \\
\text{relativistic velocity:} \quad & U^\mu = \frac{dX^\mu}{d\tau} = \gamma \frac{dX^\mu}{dt} = \gamma\,(c, u_x, u_y, u_z) \\
\text{relativistic energy-momentum:} \quad & P^\mu = mU^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right), \quad E = \gamma mc^2 \\
\text{relativistic current density:} \quad & J^\mu = (c\rho, j_x, j_y, j_z), \quad \rho = \text{ electric charge density}, \\
& \qquad\qquad\qquad\qquad\qquad\qquad \vec{j} = \text{ electric current density}
\end{aligned}$$

Each of these should be thought of as a column vector with 4 rows. Here we are using uppercase quantities with Greek indices to denote 4-vectors, and lowercase quantities to denote spatial 3-vectors; for instance, we might use a shorthand $P^\mu = (E/c, \vec{p})$.

A 4-vector V transforms under boosts by Lorentz transformations, as $V \to \Lambda V$. For example, consider the velocity 4-vector U for a particle that is moving with speed $\vec{u}$ in frame F. In a frame F' moving with speed v in the horizontal direction, the 4-velocity is transformed as

$$U' = \Lambda U = \begin{pmatrix} \gamma_v & -\gamma_v \frac{v}{c} & 0 & 0 \\ -\gamma_v \frac{v}{c} & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_u c \\ \gamma_u u_x \\ \gamma_u u_y \\ \gamma_u u_z \end{pmatrix} = \begin{pmatrix} \gamma_v \gamma_u \left(c - \frac{v}{c} u_x\right) \\ \gamma_u \gamma_v \left(-v + u_x\right) \\ \gamma_u u_y \\ \gamma_{u'} u_z \end{pmatrix} = \begin{pmatrix} \gamma_{u'} c \\ \gamma_{u'} u'_x \\ \gamma_{u'} u'_y \\ \gamma_{u'} u'_z \end{pmatrix}$$

where γ_u is the γ-factor that is a function of the velocity of the particle, $\gamma_u = 1/\sqrt{1 - \frac{\vec{u}^2}{c^2}}$, and γ_v is the γ-factor that boosts between frames related by relative velocity v, $\gamma_v = 1/\sqrt{1 - \frac{v^2}{c^2}}$. We can solve this system of equations for the velocity in the primed frame as follows. The first equation allows us to solve for the γ-factor in the primed frame,

$$\gamma_{u'} c = \gamma_v \gamma_u \left(c - \frac{v}{c} u_x\right) \quad \Rightarrow \quad \gamma_{u'} = \gamma_v \gamma_u \left(1 - \frac{v u_x}{c^2}\right)$$

Then, the second, third, and fourth equations simplify as:

$$\begin{aligned}
u'_x &= \frac{\gamma_u \gamma_v}{\gamma_{u'}} (u_x - v) = \frac{u_x - v}{1 - \frac{v u_x}{c^2}} \\
u'_y &= \frac{\gamma_u}{\gamma_{u'}} u_y = \frac{u_y}{\gamma_v \left(1 - \frac{v u_x}{c^2}\right)} \\
u'_z &= \frac{\gamma_u}{\gamma_{u'}} u_z = \frac{u_z}{\gamma_v \left(1 - \frac{v u_x}{c^2}\right)}
\end{aligned}$$

which we recognize as precisely the Lorentz velocity addition formulas (see (3.14)).

Furthermore, the inner products of each of these 4-vectors with themselves defines a quantity that is invariant under Lorentz transformations: for instance,[11]

$$\begin{aligned} \Delta X \cdot \Delta X &= (\Delta s)^2 \qquad \text{the invariant interval} \\ U \cdot U &= -c^2 \qquad \text{the speed of light} \\ P \cdot P &= -(mc)^2 \quad \text{energy momentum conservation} \end{aligned} \tag{3.28}$$

where we need to keep in mind that the inner product has a factor of the metric η hidden inside it; $P \cdot P \equiv P^T \eta P$, $U \cdot U \equiv U^T \eta U$, *etc.* The right-hand-side of these expressions are all quantities that you've already learned are measured the same in all inertial frames: the value of the invariant interval; the speed of light; the rest energy of the particle mc^2.

Actually, the situation is *even better* than this. Call an object with some number of spacetime indices $\mu, \nu, \ldots$ that run over $0, 1, 2, 3$ a **Lorentz tensor**. For example, any column 4-vector is a single-index Lorentz tensor, while the metric $\eta_{\mu\nu}$ and Lorentz transformation $\Lambda_{\mu\nu}$ are examples of two-index Lorentz tensors. (Switching from the column vector, matrix nomenclature to this "tensor" nomenclature is useful because there are also examples of Lorentz tensors with even more indices, which are kind of like higher-dimensional versions of matrices.) The statement is: any time I can take inner products of Lorentz tensors in such a way that all indices are contracted, so that the final outcome is a Lorentz *scalar* with no indices, that scalar is guaranteed to be invariant under Lorentz transformations: all observers in different inertial reference frames related by boosts will measure exactly the same value for that scalar. This is true for the inner products we wrote in (3.28), but it is also true for any other scalar object you can construct out of Lorentz tensors.[12]

Conservation of electric charge For example, consider the electric current 4-vector J^μ, whose components are ρ (the charge density that satisfies $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ in Gauss' law) and $\vec{j}$ (the electric current density that satisfies $\vec{\nabla} \times \vec{B} = \mu_0(\vec{j} + \epsilon_0 \partial_t \vec{E})$ in Ampere's law). Taking the 4-divergence of this 4-vector will always yields an invariant scalar, which one can check is actually always equal to 0 in any frame of reference:

$$\partial \cdot J = 0 \tag{3.29}$$

Explicitly, this equation says:

$$\partial \cdot J \equiv (\partial)^T \eta J = \begin{pmatrix} -\frac{1}{c}\partial_t & \partial_x & \partial_y & \partial_z \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\rho \\ j_x \\ j_y \\ j_z \end{pmatrix} = \partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0$$

This is literally the equation that states that electric charge is conserved in electromagnetism: the rate of change of charge through a fixed volume is equal to the net current flowing through the boundary of that volume; in more familiar integral form,

$$\frac{d}{dt} \int_V \rho \, dV = - \int_{\partial V} \vec{j} \cdot d\vec{s}$$

[11] Apologies, I left off the minus signs on two of these expressions in lecture!

[12] The general statement is that Lorentz tensors transform under Lorentz transformations so that each Lorentz index $\nu = \{0, 1, 2, 3\}$ is contracted with a Lorentz transformation matrix $\Lambda_{\mu\nu}$. To be careful about index contraction for higher-index tensors it is useful to define a notion of "lower" and "upper" indices that can be raised / lowered by the metric, to keep track of when we need to insert η in summing over the indices. Since in these notes we have only needed to learn how to act on column 4-vectors and take inner-products (we haven't yet encountered any 2- or higher index tensors that actually transform under Lorentz transformations (although see (3.33)) – note that η is invariant under Lorentz transformations), we haven't bothered with this extra complication.

Conservation of 4-momentum A nice application of this idea is to conservation of energy and momentum in special relativity. In Physics 200, you learned that the relativistic momentum of an object of rest mass m and velocity $\vec{u}$ that is conserved in all inertial frames is $\vec{p} = \gamma_u m\vec{u}$. You also learned the extremely useful equation that by combining the definitions of a particle's relativistic energy $E = \gamma_u mc^2$ and momentum $\vec{p} = \gamma_u m\vec{u}$, they satisfy the relation

$$E^2 = p^2c^2 + m^2c^4$$

where $p^2 = \vec{p} \cdot \vec{p} = p_x^2 + p_y^2 + p_z^2$. Now we can understand that this equation follows naturally from the invariant inner product of 4-momentum. The 4-momentum of a massive particle that is traveling with relativistic 4-velocity U is given by

$$P^\mu = mU^\mu = m\gamma\,(c, u_x, u_y, u_z) = \left(\frac{E}{c}, p_x, p_y, p_z\right)$$

where we have equated the timelike component P^0 of the particle's momentum with its total relativistic energy E/c, so that indeed with this definition $E = \gamma mc^2$ (and of course $p_x = \gamma m u_x$, and so on). The conservation of relativistic energy and momentum is now packaged together into the conservation of energy-momentum, via conservation of the 4-vector P. Under a Lorentz transformation, $P \to P' = \Lambda P$, and energy-momentum is conserved in the primed frame just as well as in the unprimed frame.

Taking the dot product of P with itself yields

$$P \cdot P = P^T \eta P = \begin{pmatrix} \frac{E}{c} & p_x & p_y & p_z \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{E}{c} \\ p_x \\ p_y \\ p_z \end{pmatrix} = -\frac{E^2}{c^2} + \vec{p}^2\,.$$

We have claimed that this should equal a quantity that is invariant under Lorentz transformations, so is the same in all frames. Therefore, we might as well compute it in the rest frame of the particle $P = (mc, 0, 0, 0)$. In this frame,

$$P \cdot P = -m^2c^2$$

But since the inner product is invariant, we can equate it to the general equation valid in any frame,

$$-m^2c^2 = -\frac{E^2}{c^2} + \vec{p}^2 \quad \Rightarrow \quad E^2 = \vec{p}^2c^2 + m^2c^4$$

reproducing the frame-invariant relation you learned about in Physics 200. This is the fundamental equation that tells us that in relativity, mass can be converted into energy and vice versa.

What about massless particles? Massless particles evidently obey $P \cdot P = 0$, so that their 4-momentum is *null* (along the path of a light ray). Here we are using the same definition of null that we used previously: a null vector V satisfies $V \cdot V = 0$, much as a null coordinate vector X satisfies $dX \cdot dX = ds^2 = 0$. This fact clarifies one of our original postulates of special relativity, that the speed of light is the same for all inertial frames. Why is the propagation of light singled out for special treatment? The answer is that the photon – the particle that makes up light – is massless. Any massless particle must travel at the speed of light, obeying the dispersion relation $E = pc$, while all massive particles must go slower.

Exercise 3.10

Photons are massless particles whose 4-momentum will generally take the form

$$P^\mu = E/c\,(1, \hat{\vec{p}})$$

where $\hat{\vec{p}}$ is a unit vector in the direction of the particle's motion. (This definition is constructed to satisfy $P \cdot P = 0$ for a massless particle, since $\hat{\vec{p}} \cdot \hat{\vec{p}} = 1$). We also know from quantum mechanics that a photon has energy $E = hc/\lambda$, for λ the wavelength of the associated electromagnetic wave.

Let λ be the wavelength of the light in the frame where the light source is at rest. By acting with a Lorentz transformation on P, determine the wavelength λ' that an observer in a frame F' moving with speed v away from the light source will observe. You may assume that both the light and the relative speed v are directed along the x-axis.

——— *End Lecture 12.*

The relation between symmetry and conservation As a comment, we have seen that by understanding the isometries of Minkowski spacetime and developing special relativity in a way that maximally utilizes these symmetries, it is straightforward to identify conservation laws. The general statement (which follows from Noether's theorem, to be discussed in the next section first in the context of classical mechanics) is that for every continuous symmetry of a system, there is an associated conserved quantity. In the case of special relativity, there are 10 continuous spacetime symmetries corresponding to the 10 parameters of the Poincaré group. The associated 10 conserved quantities are: angular momentum (3, related to spatial rotations), spatial momentum (3, related to spatial translations), energy (1, related to time translations), and the three components of the velocity of the center of mass (3, related to boosts). We will return to this point of view in the context of Noether's theorem.

3.7 Covariant formulations of classical mechanics and electromagnetism

We will end this section on special relativity with some discussion for how Newtonian physics and electromagnetism can be formulated in a way that is compatible with special relativity.

Newton's law in covariant form The centerpiece of pre-relativity classical physics is Newton's second law, $\vec{F} = m\vec{a}$. We have already seen that this law holds for Galilean relativity, but is not invariant under Lorentz transformations. One should expect that the analogous law that holds in special relativity should involve 4-vectors, and indeed, we are now in a position to write the relativistically-compatible version of Newton's second law:

$$F^\mu = \frac{dP^\mu}{d\tau} = \gamma \frac{dP^\mu}{dt} \tag{3.30}$$

where F^μ are the components of a 4-vector force, defined in terms of the standard 3-spatial components $\vec{f}$ of the force as:

$$F^\mu = \begin{pmatrix} F^0 \\ \gamma f_x \\ \gamma f_y \\ \gamma f_z \end{pmatrix}$$

With this definition, we see that the spatial components of Newton's equation actually agree with the form that we're used to: in a frame F, $\frac{d\vec{p}}{dt} = \vec{f}$, where $\vec{p}$ is the relativistic spatial momentum. Meanwhile, the time component is related to the rate of change of energy with time:

$$F^0 = \gamma \frac{dP^0}{dt} = \frac{\gamma}{c}\frac{dE}{dt}\,. \tag{3.31}$$

A first pass at covariant electromagnetism The reason that (3.30) is rarely used in relativistic physics is because most of the forces we come across in classical physics (including gravity!) are not valid in the context of special relativity where all frames must be inertial – they require the formalism of general relativity. An exception to this rule is the electromagnetic force. This is because Maxwell's laws governing the behavior of electric and magnetic fields are already compatible with special relativity.

Consider a particle of electric charge q moving with velocity $\vec{u}$ in some inertial frame of reference. The electromagnetic force on this particle is given by the so-called Lorentz force, $\vec{f} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$. The relativistic formulation of this force arises from identifying the following electromagnetic 4-vector force:

$$F = -qG\eta U \quad \leftrightarrow \quad F^\mu = -q\sum_{\nu,\rho=0}^{3} G_{\mu\nu}\eta_{\nu\rho}U^\rho \tag{3.32}$$

where U is the 4-velocity of the particle, η is the metric, and $G_{\mu\nu}$ is a 2-index tensor (4×4 matrix) called the **electromagnetic field strength tensor**, which contains the electric and magnetic fields:

$$G_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}$$

Doing out the matrix multiplication in (3.32), the expression reads:

$$F^\mu = \begin{pmatrix} F^0 \\ \gamma\vec{f} \end{pmatrix} = \begin{pmatrix} \frac{q\gamma}{c}\vec{E}\cdot\vec{u} \\ q\gamma\left(\vec{E} + (\vec{u}\times\vec{B})\right) \end{pmatrix}$$

The spatial components of this equation yield precisely the spatial components of the 3-dimensional Lorentz force, $\vec{f} = q\left(\vec{E} + \vec{u} \times \vec{B}\right)$. Meanwhile, the time component gives the rate of work done: using (3.31), we identify

$$F^0 = \frac{\gamma}{c}\frac{dE}{dt} = \frac{q\gamma}{c}\vec{E}\cdot\vec{u} \quad \Rightarrow \quad \frac{dE}{dt} = q\vec{E}\cdot\vec{u}$$

which is indeed the power.

The electromagnetic field strength tensor $G_{\mu\nu}$ (usually conventionally called $F_{\mu\nu}$, but we are using the letter G to not confuse it with the force vector that we are denoting with an F) is the nice, Lorentz-covariant way to package the electric and magnetic fields into one object. This is necessary because in different frames of reference, charges that appear stationary in one frame will be moving in another, so that the electric and magnetic fields generated by the particle will be different! By packaging everything into a tensor, there is a straightforward algorithm for how to transform the E and B fields in one frame to those as measured in another frame,

by acting on the tensor $G_{\mu\nu}$ with Lorentz transformations. The rule for Lorentz-transforming a 2-tensor like $G_{\mu\nu}$ between frames is as follows:

$$G' = \Lambda G \Lambda^T \tag{3.33}$$

We need 2 copies of the Lorentz transformation, one for each index. Note that this rule is compatible with our identification of the force 4-vector (3.32); since G as a 2-tensor transforms as $G \to \Lambda G \Lambda^T$ and U as a 4-vector transforms as $U \to \Lambda U$, the right-hand-side transforms as

$$G\eta U \to (\Lambda G \Lambda^T)\eta(\Lambda U) = \Lambda G(\Lambda^T \eta \Lambda)U = \Lambda(G\eta U)$$

where we used the group property $\Lambda^T \eta \Lambda = \eta$. Therefore, this whole expression just transforms like a 4-vector with a single factor of Λ, as it must since it is identified with a 4-vector F that transforms $F \to \Lambda F$.

Furthermore, Maxwell's equations can be very simply written in a Lorentz-covariant way with this notation. For instance, 2 of the 4 of them are encapsulated by the matrix equations:

$$(\partial)^T \eta G = \mu_0 J^T \quad \leftrightarrow \quad \partial \cdot G = \mu_0 J^T \tag{3.34}$$

In the following exercise, you will demonstrate that this matrix equation indeed yields Maxwell's equations.

Exercise 3.11

(a) Use the Lorentz transformation law (3.33) for the electromagnetic field strength tensor to derive the electric and magnetic field components $\vec{E}'$ and $\vec{B}'$ in a primed frame moving with speed v in the $+\hat{x}$ direction relative to the unprimed frame.

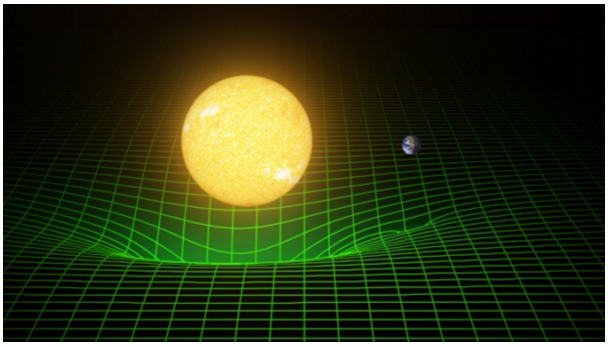
(b) Show that two of Maxwell's equations follow from $(\partial)^T \eta G = \mu_0 J^T$.

3.8 An introduction to General Relativity

- Richard Feynman gives a nice conceptual introduction to General Relativity in these lectures.
- If you are interested in learning more about gravitational waves, I recommend Cambridge Professor David Tong's lecture notes on general relativity, especially section 5 "When Gravity is Weak" where he goes into some detail on the wave solutions to Einstein's equations both in vacuum and in response to an accelerating mass.

The consistency of special relativity relied on the existence of inertial frames of reference: frames in which a particle at rest in it stays at rest if no force acts on it. In Newtonian mechanics, gravity is regarded as a force. However, it turns out that gravity is distinguished from all the other forces in a very important way: *all* massive particles, regardless of their internal composition, follow exactly the same trajectory in a gravitational field. What this means for us as experimentalists on Earth is that we can't experimentally find the trajectory of a particle that's unaffected by gravity, and so can't define an inertial frame at rest on the Earth.

Consider the following example: imagine yourself on a uniformly accelerating rocket ship in empty space, where there is no gravity. From your point of view inside the rocket, there is a gravitational field inside the rocket: if I drop a ball, it accelerates downwards, much like



dropping a ball on Earth. Just standing stationary in the ship, I have a weight equal to the force required to keep myself accelerating along with the ship that's proportional to my mass—just as I do on Earth.

In fact, any experiment I could do in such a uniformly accelerating reference frame of acceleration a is completely indistinguishable from the same experiments performed in a non-accelerating reference frame situated in a uniform gravitational field in which the acceleration due to gravity is $g = -a$. This is the *(weak) principle of equivalence* between gravity and acceleration. Another way to put this is that an object's *inertial mass*—the mass that measures resistance to being accelerated by a force, $m_i = F/a$— and *gravitational mass*—the degree to which it's affected by a gravitational field, $m_g = F_g/g$—are the same. This observation, made by Einstein in 1915, is the starting point for the theory of **general relativity**. General relativity reduces to special relativity in the limit that gravitational effects become less important.

Einstein developed an ingenious way to deal with these accelerating frames: by replacing the concept of gravitational force with the curvature of spacetime. The presence of a mass causes a curvature of spacetime in the vicinity of the mass, and this curvature dictates the path in spacetime that all objects must follow. As summarized by American theoretical physicist John Wheeler, "Space tells matter how to move and matter tells space how to curve." General relativity consists of two parts: how matter affects spacetime and how spacetime affects matter.

In particular, the presence of mass leads to a nontrivial metric g of spacetime, so that the invariant interval satisfies

$$ds^2 = dX \cdot dX = (dX)^T g(dX)$$

Locally space should look like Minkowski space so that we recover special relativity. This means that if we zoom in to very tiny distances, $g = \eta$ so that the metric is approximately that of special relativity, with differences being second order in the coordinates. More generally, g can be quite complicated.

Einstein derived the equations of motion that tell us how the presence of matter allows us to solve for the metric of spacetime g, encapsulating how the curvature of spacetime reacts to matter. *Einstein's field equations* take the form,

$$\widetilde{G}_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{3.35}$$

where $\widetilde{G}_{\mu\nu}$ is a 2-tensor that is a very complicated, quartic, function of the metric g involving two derivatives of the metric (the tilde is meant to distinguish it from the field strength tensor $G_{\mu\nu}$ we introduced for electromagnetism); $T_{\mu\nu}$ is a 2-tensor that encapsulates the effect of the gravitational potential due to matter; and G is Newton's gravitational constant. The field equations (3.35) then constitute a set of complicated second order differential equations that can

be solved for the metric. Once the metric is solved for, one can compute the paths of particles in that spacetime by integrating up ds along a path. Particles will want to follow geodesics in the curved spacetime, so the dynamics of particles follows from solving for the geodesics in a spacetime with metric g. Of course, in the dynamical problem everything is time dependent – as particles move, they tell spacetime how to curve according to (3.35), which then determines the geodesics of the particles, and so on, in a feedback loop – so this is quite a complicated system of equations! General relativists often rely heavily on numerics to solve these systems of equations for the dynamics of stars, galaxies, *etc.*

Black Holes One famous example is the *Schwarzschild metric*, which describes the space outside of a spherically symmetric object of mass M (for example, a star, or a black hole) by the following invariant distance interval,

$$ds^2 = -\left(1 - \frac{2GM}{rc^2}\right) c^2 dt^2 + \left(1 - \frac{2GM}{rc^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

Here r is the radius away from the object and G, c are constants, so this solution depends on the single parameter M. This solution follows solving (3.35) for g in the presence of a spherically symmetric object of mass M sitting at the center of the coordinate system.

This is actually the unique spherically symmetric, asymptotically (at $r \to \infty$) flat solution to the vacuum Einstein equations. In particular, far away from the object at $r \to \infty$, this metric coincides with the usual Minkowski metric written in spherical polar coordinates:

$$ds^2 \to -c^2 dt^2 + dr^2 + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

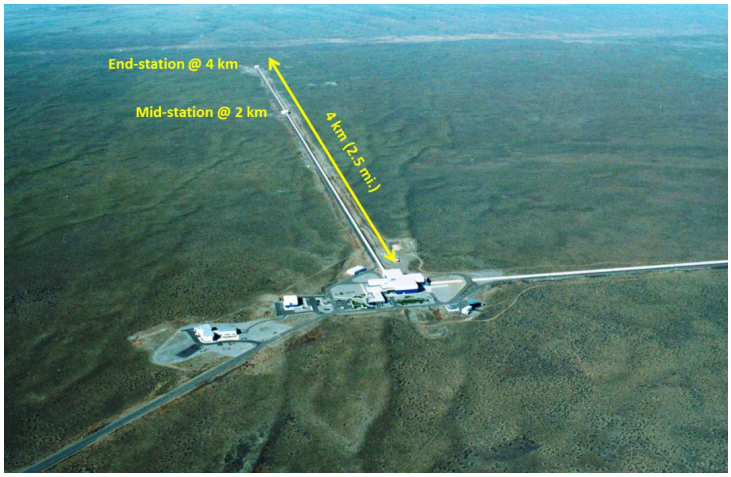
Closer to the star this metric describes how the space around a spherically symmetric object warps.

There are two values of r where the Schwarzschild metric appears to go bad: it is singular at $r = 0$ and at $r = \frac{2GM}{c^2}$. The radius $r = \frac{2GM}{c^2}$ where the radial component of the metric diverges is known as the Schwarzschild radius, and corresponds to the event horizon of a black hole. While this radius is special for many reasons, it turns out that this is just a coordinate singularity and not a true singularity of the spacetime, in the sense that a change of coordinates gets rid of the singular behavior of the metric in this region. On the other hand, the $r \to 0$ divergence is a true spacetime singularity, and a genuine feature of the black hole.

Gravitational waves The metric is not a static thing – it reacts to the presence of matter, changing with time as matter moves throughout spacetime. One consequence of this is the existence of *gravitational waves*. Massive accelerating objects disrupt spacetime in such a way that ripples can propagate over long distances.

In a truly incredible feat of engineering decades in the making, LIGO (Laser Interferometer Gravitational-wave Observatory) made the first detection of gravitational waves in 2015, for which three members of the collaboration were awarded the 2017 Novel prize. They detected tiny waves that were generated by two colliding black holes 1.3 billion light-years away. 1.3 billion light years ago (at roughly the time multicellular life was forming on Earth!) these black holes collided, emitting an energy equivalent to three times the mass of the sun in the process that was sent out in ripples of spacetime.

Gravitational waves are sourced by some event, but then propagate in the vacuum, which amounts to solving Einstein's equation (3.35) with 0 on the right-hand-side: $\widetilde{G}_{\mu\nu} = 0$. We



decompose the metric into the flat Minkowski metric plus a small perturbation meant to describe the ripple,

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}\,, \qquad |h| \ll 1$$

We can describe the equation of motion obeyed by the perturbations h by expanding out Einstein's field equations to first order in h (yielding the "linearized" Einstein's equations). In this case the equation greatly simplifies to

$$(\partial \cdot \partial)\left(h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu}\right) = 0$$

This is a wave equation – it's the 4-dimensional Laplacian $-\partial_t^2 + \nabla^2$ acting on some field $\overline{h} =$ (the combination of the perturbation h and metric η in parenthesis), and there are plane-wave like solutions to it of the form:

$$\overline{h}_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}h\eta_{\mu\nu} = \mathrm{Re}\left(H_{\mu\nu}e^{-i\omega t + i\vec{k}\cdot x}\right)$$

where $H_{\mu\nu}$ is a polarization matrix, and the wave propagates at the speed of light c. A general gravitational wave solution will consist of superpositions of these plane waves.

——— *End Lecture 13.*

4 Classical Mechanics and Conservation Laws

Possibly helpful resources:

- **MOST RECOMMENDED:** *A First Course on Symmetry, Special Relativity, and Quantum Mechanics* by Kunstatter and Das, Section 3.6 on Variational Mechanics and Noether's theorem.
- University of Cambridge Professor David Tong's lecture notes on Classical Dynamics are a nice additional resource, especially parts of Sections 2 and 4. Available at this link.

4.1 Lagrangian Mechanics and the Principle of Least Action

Let us start by recalling the definition of the Lagrangian. Suppose we have a system of N particles in one dimension, with coordinates x^A where $A = 1, \ldots, N$. (This discussion readily generalizes to three dimensions, where we could take $A = 1, \ldots, 3N$.) The particles have masses m_A and evolve in some potential $U(x)$. Newton's equations relate the derivative of the particles' momenta to the gradient of the potential as

$$p_A = m_A \dot{x}^A \,, \qquad \dot{p}_A = -\frac{\partial V}{\partial x^A}$$

where the dot denotes differentiation with respect to time, $\dot{x} = dx/dt$. The *Lagrangian* $\mathcal{L}$ is a function of the positions x^A and velocities $\dot{x}^A$ of the particles, given by their kinetic energy $K(\dot{x}^A) = \frac{1}{2}\sum_A m_A(\dot{x}^A)^2$, minus their potential energy $U(x^A)$:

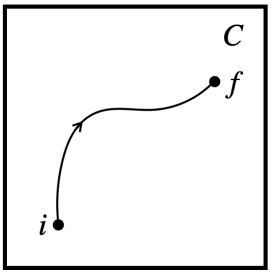
$$\mathcal{L}(x^A, \dot{x}^A) = K(\dot{x}^A) - V(x^A) = \frac{1}{2}\sum_A m_A(\dot{x}^A)^2 - U(x^A) \tag{4.1}$$

The coordinates $x^A(t)$ parameterize an N-dimensional *configuration space*, C. (For a d-dimensional system, there will be dN degrees of freedom and thus dN coordinates parameterizing the configuration space – but for now lets stick with $d = 1$.) A point in the configuration space specifies a particular configuration of the system, namely, the positions of all N particles at some moment in time. Time evolution is some curve, or path, in configuration space.

Suppose we know the initial positions of the particles at some initial time t_i, and the final positions at some final time t_f. We consider all smooth paths $x^A(t)$ with these endpoints fixed:

$$x^A(t_i) = x^A_{\text{initial}} \,, \qquad x^A(t_f) = x^A_{\text{final}}$$

Of course, there the system only takes one path which evolves according to the equations of motion from x^A_{initial} to x^A_{final}. Which path does the system take?



The answer is given by the following prescription. To each path $x^A(t)$, we assign a number called the *action*:

$$S[x^A(t)] = \int_{t_i}^{t_f} \mathcal{L}\left(x^A(t), \dot{x}^A(t)\right)\, dt \tag{4.2}$$

In other words, take the Lagrangian (a function of the path $x^A(t)$ and its derivatives) evaluated on that path, and integrate it between the initial and final times. The output is a number, the action S. The action is therefore what we call a *functional*: an object which takes a function as input and as output spits out a number, versus a function which takes a number as input and spits out a number. The actual path of the system is given by the following theorem:

Theorem (Principle of Least Action): The path taken by the system is an extremum of the action S, defined as in (4.2).

The principle of least action is one of the most important results in physics. At first glance it seems highly non-intuitive, so let's prove it. We can consider a given path $x^A(t)$ between the fixed endpoints, and varying it slightly while leaving the endpoints fixed,

$$x^A(t) \to x^A(t) + \delta x^A(t)\,, \qquad \delta x^A(t_i) = \delta x^A(t_f) = 0$$

We will first show that the path which extremizes the action under such a variation satisifies the Euler-Lagrange equations. The change in the action is computed as:

$$\begin{aligned}
\delta S &= \delta\left[\int_{t_i}^{t_f} \mathcal{L}\, dt\right] \\
&= \int_{t_i}^{t_f} \delta\mathcal{L}\, dt \qquad\qquad\qquad\qquad (\text{since } \delta x^A(t_i) = \delta x^A(t_f) = 0) \\
&= \int_{t_i}^{t_f} \sum_A \left(\frac{\partial\mathcal{L}}{\partial x^A}\delta x^A + \frac{\partial\mathcal{L}}{\partial\dot{x}^A}\delta\dot{x}^A\right) dt \qquad (\text{chain rule})
\end{aligned}$$

The second terms can be rewritten in such a way that we can integrate them by parts,

$$\begin{aligned}
\int_{t_i}^{t_f} \frac{\partial\mathcal{L}}{\partial\dot{x}^A}\frac{d}{dt}(\delta x^A)\, dt = \int_{t_i}^{t_f} \frac{\partial\mathcal{L}}{\partial\dot{x}^A} d(\delta x^A) &= \frac{\partial\mathcal{L}}{\partial\dot{x}^A}\delta x^A\bigg|_{t_i}^{t_f} - \int_{t_i}^{t_f} \delta x^A d\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^A}\right) \\
&= \frac{\partial\mathcal{L}}{\partial\dot{x}^A}\delta x^A\bigg|_{t_i}^{t_f} - \int_{t_i}^{t_f} \delta x^A \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^A}\right) dt
\end{aligned}$$

but the boundary terms vanish, since $\delta x^A(t_i) = \delta x^A(t_f) = 0$. Substituting into δS, the expression then simplifies to

$$\delta S = \int_{t_i}^{t_f} \sum_A \left(\frac{\partial\mathcal{L}}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^A}\right)\right)\delta x^A\, dt$$

The action is an extremum if $\delta S = 0$ for all changes in the path $\delta x^A(t)$, which holds if and only if the integrand is itself zero:

$$\frac{\partial\mathcal{L}}{\partial x^A} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^A}\right) = 0 \qquad \text{for each } A = 1, \ldots, N\,. \tag{4.3}$$

These are the *Euler-Lagrange equations.*

To finish the proof, we need to show that these equations are the same as the equations of motion of the system. We can explicitly verify from (4.1) that:

$$\frac{\partial \mathcal{L}}{\partial x^A} = -\frac{\partial U(x^A)}{\partial x^A}\,,$$
$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^A}\right) = \frac{d}{dt}\left(m_A \dot{x}^A\right) = \dot{p}_A$$

so that indeed, (4.3) exactly produces Newton's equations, $\dot{p}_A = -\partial U/\partial x^A$.

Some comments:

- This is an example of a variational principle, another example of which you saw around (3.10) when computing geodesics in some space. There we saw that applying the calculus of variations to a path of length L with fixed endpoints, $L = \int_{P_i}^{P_f} ds = \int_{t_i}^{t_f} (ds/dt)\, dt$, tells you that the solution to the Euler-Lagrange equations with $\mathcal{L}$ replaced by the integrand ds/dt extremizes the length of the path (the "action" in this case). Same concept here, except the quantity we're extremizing is S, and the integrand is the Lagrangian $\mathcal{L}$.
- Does $\delta S = 0$ correspond to a minimum, maximum, or saddle point? Since $\mathcal{L} = K - U$, we can always make S bigger by considering a very fast path with large kinetic energy (S is not bounded from above!), so the true path is never a maximum. Then, the path is in general either a minimum or a saddle point of S.

Exercise 4.1

By the variation δS we really mean

$$\delta S \equiv S\left[x + \delta x(t)\right] - S[x(t)]$$

Explicitly compute the expression on the right for the Lagrangian of a single particle with kinetic energy $\frac{1}{2}m\dot{x}^2$ moving in a potential $U(x)$, only keeping terms that are first order in the small change δx (namely, neglecting terms of order $(\delta x(t))^2$ and higher in the Taylor expansion). Show that setting $\delta S = 0$ leads to Newton's equation for a particle moving in a 1d potential.

Hint: you might find the discussion in section 3.6.1 of Kunstatter-Das useful.

We just showed that Newton's equations are the same as the Euler-Lagrange equations, so what did we gain by working with Lagrangian formalism? Practically speaking, we really gained two useful features. Firstly, the Euler-Lagrange equations hold in *any* coordinate system. Secondly, it is much easier to deal with constraints in the Lagrangian formalism. For complicated systems with many degrees of freedom and possibly with constraints (like a bead forced to move along a wire, or a mass attached to a rope), all one needs to do is write the Lagrangian in whatever the most convenient coordinates are, and thereby obtain the Euler-Lagrange equations of motion.

We can demonstrate the first point as follows. Let's say we change coordinates from $x_A \to q_a$,

$$q_a = q_a(x_1, \ldots, x_N, t) \tag{4.4}$$

We call the q_a *generalized coordinates*; they can be any set of independent coordinates that completely specify the system. For example, you might have in mind (in 2d) changing each particle's coordinates from Cartesian to radial coordinates, so that for each particle $r = \sqrt{x^2 + y^2}$,

and $\theta = \arctan(y/x)$. Or, you might have in mind a general Euclidean transformation of the form $x \to q = Ox + a$. You might even have in mind some coordinate change which also involves time, for example by changing to a coordinate system that itself rotates in time. (This would be a non-inertial frame!) In any case, by the chain rule we can write that for each coordinate q_a, the time derivative $\dot{q}_a$ is given by,

$$\dot{q}_a = \frac{dq_a}{dt} = \sum_A \frac{\partial q_a}{\partial x^A}\dot{x}^A + \frac{\partial q_a}{\partial t} \tag{4.5}$$

or inverting,

$$\dot{x}^A = \sum_a \frac{\partial x^A}{\partial q_a}\dot{q}_a + \frac{\partial x^A}{\partial t} \tag{4.6}$$

Let us now examine the Euler-Lagrange equations in these new coordinates. We have that

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial q_a} &= \sum_A \left(\frac{\partial \mathcal{L}}{\partial x_A}\frac{\partial x^A}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\frac{\partial \dot{x}^A}{\partial q_a} \right) \\ &= \sum_A \left(\frac{\partial \mathcal{L}}{\partial x_A}\frac{\partial x^A}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\frac{\partial}{\partial q_a}\left(\sum_b \frac{\partial x^A}{\partial q_b}\dot{q}_b + \frac{\partial x^A}{\partial t} \right) \right)\end{aligned}$$

Meanwhile, the derivative of the Lagrangian with respect to $\dot{q}_a$ is,

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_a} = \sum_A \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\frac{\partial \dot{x}^A}{\partial \dot{q}_a}$$

($\partial x^A/\partial \dot{q}_a$ is zero since we assume the coordinate change does not involve the velocities, so there is only one set of terms in the chain rule for this case.) It is a useful fact that

$$\frac{\partial \dot{x}^A}{\partial \dot{q}_a} = \frac{\partial x^A}{\partial q_a},$$

or in other words that we can "cancel the dots" so to speak (verify this for yourself using (4.6)!). Therefore, we can compute:

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) &= \frac{d}{dt}\left(\sum_A \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\frac{\partial x^A}{\partial q_a} \right) \\ &= \sum_A \left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^A} \right)\frac{\partial x^A}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\left(\frac{d}{dt}\frac{\partial x^A}{\partial q_a} \right) \right] \\ &= \sum_A \left[\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^A} \right)\frac{\partial x^A}{\partial q_a} + \frac{\partial \mathcal{L}}{\partial \dot{x}^A}\left(\sum_b \frac{\partial^2 x^A}{\partial q_a \partial q_b}\dot{q}_b + \frac{\partial x^A}{\partial^2 q_a \partial t} \right) \right]\end{aligned}$$

where we used (4.6) in the last line. Finally, we can evaluate the Euler-Lagrange equations in the q coordinates. The terms involving the sum over b cancel, and we are left with:

$$\frac{\partial \mathcal{L}}{\partial q_a} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}_a}\right) = \sum_A \left[\frac{\partial \mathcal{L}}{\partial x_A} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^A} \right) \right]\frac{\partial x^A}{\partial q_a}$$

This result says that if the Euler-Lagrange equations hold in the x^A coordinate system (so that everything in the $[\dots]$ on the right-hand-side vanishes), then they *also* necessarily hold for every index a in the q_a coordinate system (so that the left-hand-side vanishes). This proof holds as long as our choice of coordinates is invertible, so that we can invert $q_a = q_a(x_A, t)$ to

$x^A = x^A(q_a, t)$, which is true as long as the Jacobian of the transformation $\det(\partial x^A/\partial q_a)$ is nonzero.

In addition to these practical reasons, there are two additional very good reasons to study Lagrangian mechanics.

- All the fundamental laws of physics can be written in terms of an action principle, including electromagnetism, special relativity, general relativity, and the standard model of particle physics. For now we are focusing on classical Lagrangian mechanics, but later we will discuss generalizations to some of these other cases.
- The Lagrangian formalism makes symmetries manifest! We will explore this idea in the next few sections.

——— *End Lecture 14.*

4.2 The free particle Lagrangian from symmetry principles

Let us return to the general form of the Lagrangian for a particle of mass m,

$$\mathcal{L} = K - U = \frac{1}{2}m\dot{\vec{x}}^2 - U(\vec{x})\,.$$

For a free particle in three dimensions, $U = 0$ and $\vec{x} = (x, y, z)$. You might wonder: can we argue based on general principles that this is the free particle Lagrangian? Do we need to already have known Newton's laws? Why does the Lagrangian go like v^2 rather than some other power of v?

The answer is due to the symmetries of space and time, along with the assumption of the least action principle. To see this, let us work (for simplicity) in Cartesian coordinates $\vec{x} = (x, y, z)$. We make the following assumptions:

- *Assumption 1:* For a free particle, all points in space are equal (*i.e.* space is *homogeneous*). In other words, the Lagrangian can't depend on *where*, the place from which we measure its progress: $\mathcal{L}(\vec{x}, \dot{\vec{x}}, t) \to \mathcal{L}(\dot{\vec{x}}, t)$. This is to say that the Lagrangian should be translation invariant.
- *Assumption 2:* For a free particle, all points in time are equal (*i.e.* time is *homogeneous*). In other words, the Lagrangian can't depend on *when*, the time at which we start our measurement: $\mathcal{L}(\dot{\vec{x}}, t) \to \mathcal{L}(\dot{\vec{x}})$. This is to say the Lagrangian should be time-translation invariant.
- *Assumption 3:* Space is *isotropic* (all directions are equal) – it should not depend on the orientation of the coordinate system. Therefore, $\mathcal{L}$ can depend only on the magnitude of the velocity and not its direction, $\mathcal{L}(|\dot{\vec{x}}|) = \mathcal{L}(v)$, where $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$.
- *Assumption 4:* We assume the principle of least action, so that the Euler-Lagrange equations (4.3) hold. For instance, take the x-direction, which leads to the equation of motion

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}}\right) = \frac{\partial \mathcal{L}}{\partial x}$$

By *Assumption 1*, the right-hand-side = 0, since $\mathcal{L}$ is only a function of $\dot{x}$ and not x itself. Therefore,

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \text{constant in time} \equiv c_x$$

The same is true for the other directions; we $\partial\mathcal{L}/\partial\dot{y}$ = some constant we will call c_y, and $\partial\mathcal{L}/\partial\dot{z}$ = some constant we will call c_z.

We can use this conclusion to learn about the dependence of the particle's motion on the speed v. Using the chain rule,

$$\begin{aligned}\frac{d\mathcal{L}(v)}{dv} &= \frac{\partial \mathcal{L}}{\partial \dot{x}}\frac{d\dot{x}}{dv} + \frac{\partial \mathcal{L}}{\partial \dot{y}}\frac{d\dot{y}}{dv} + \frac{\partial \mathcal{L}}{\partial \dot{z}}\frac{d\dot{z}}{dv} \\ &= c_x\frac{v}{\dot{x}} + c_y\frac{v}{\dot{y}} + c_z\frac{v}{\dot{z}}\end{aligned}$$

The only way that $\mathcal{L}$ could be a function of v only (and constants), is if $c_x \propto \dot{x}$, so that $\dot{x}$ is itself a constant; and similarly, $c_y \propto \dot{y}$ is a constant, and $c_z \propto \dot{z}$ is a constant. But if $\dot{x}, \dot{y}$, and $\dot{z}$ are all constants, then $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$ is itself a constant of motion! We conclude that not only can the Lagrangian of a free particle can only depend on the magnitude of its velocity, but that the result will be motion with *constant* velocity. This is the law of inertia: *A free particle moves with constant speed.*

- *Assumption 5:* The equations of motion should take the same form in all inertial frames of reference; namely, the principle of Newtonian / Galilean relativity should hold. (Physics should not depend on the physicist!) There is no preferred frame of reference, but instead the laws of physics should be unchanged regardless of the frame.

Let us impose this last assumption on our Lagrangian $\mathcal{L}(v)$. In order to relate measurements made in one frame of reference with those in another, we use the Galilean transformation between frames. Let's say that an observer in our original frame F tracks the free particle's motion with spatial coordinates (x, y, z) and time coordinate t, and speed $v = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$. An observer in frame F' which is moving at speed u in the positive x-direction relative to F (for example, think an observer sitting on a train that is moving with constant speed u), would track the free particle with spatial coordinates (x', y', z'), time coordinate t', and speed $v' = \sqrt{(\dot{x}')^2 + (\dot{y}')^2 + (\dot{z}')^2}$. Assuming that the origins of these coordinate axes coincide, so that $(x, y, z) = (x', y', z') = (0, 0, 0)$ at initial time $t = t' = 0$, the Galilean transformation of the coordinates between these frames is

$$x' = x - ut\,, \qquad y' = y\,, \qquad z' = z\,, \qquad t' = t.$$

so that the velocities transform

$$\dot{x}' = \dot{x} - u\,, \qquad \dot{y}' = \dot{y}\,, \qquad \dot{z}' = \dot{z} \quad \Rightarrow \quad v^2 = (v')^2 + 2\dot{x}u - u^2 = (v')^2 + 2u\dot{x}' + u^2$$

(Without loss of generality, let's actually consider $\mathcal{L}(v^2)$ in this part, so that we don't have to deal with all the square roots – this is not necessary, just useful.) Then, we can see that the Lagrangians in the two frames are related by,

$$\mathcal{L}(v^2) = \mathcal{L}((v')^2 + 2u\dot{x}' + u^2)$$

Since Galilean relativity should hold for small velocities as well as large ones, we can assume that u is small and Taylor expand around $u = 0$:

$$\begin{aligned}\mathcal{L}(v^2) &= \mathcal{L}((v')^2 + 2u\dot{x}' + u^2) \\ &= \mathcal{L}((v')^2) + \frac{\partial \mathcal{L}((v')^2)}{\partial (v')^2}(2u\dot{x}' + u^2) + \ldots\end{aligned} \tag{4.7}$$

So far so good. Now, in order that the two Lagrangians describe the same physics, and so have the same equations of motion, we need that they differ by a total derivative with respect to time:

$$\mathcal{L}(v^2) - \mathcal{L}((v')^2) = \frac{d}{dt}F \tag{4.8}$$

The easiest way to see this is to add a total derivative to the Lagrangian, and show that the variation of the action doesn't change:

$$\mathcal{L}' = \mathcal{L} + \frac{d}{dt}F \quad \Rightarrow \quad S' = S + F\Big|_{t_i}^{t_f}$$

The action changes by a boundary term, but since all trial paths pass through the same end-points t_i and t_f, this boundary term is a constant that doesn't affect the variation δS. (A straightforward but harder thing to show is that upon adding dF/dt to the Lagrangian, the Euler-Lagrange equations are unchanged.)

In fact, from (4.7) we have that

$$\begin{aligned}\mathcal{L}(v^2) - \mathcal{L}((v')^2) &= \frac{\partial \mathcal{L}((v')^2)}{\partial (v')^2}(2u\dot{x}' + u^2) \\ &= \frac{d}{dt}\left[\frac{\partial \mathcal{L}}{\partial (v')^2}(2ux' + tu^2)\right] - \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial (v')^2}\right)(2ux' + tu^2)\end{aligned}$$

For the right-hand-side to equal a total derivative in time for all small u, we need that $\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial (v')^2} = 0$ for all time, so that in particular, $\frac{\partial \mathcal{L}}{\partial (v')^2}$ is equal to a constant. This sets $\mathcal{L}(v')$ proportional to $(v')^2$.

We conclude that the free particle Lagrangian should be equal to a constant times the square of the velocity, v^2. That constant must have units of mass by dimensional analysis. Voila, the free particle Lagrangian $\mathcal{L} \propto mv^2$!

Of course, this still leaves some freedom: an overall scalar prefactor does not affect any of the above conclusions, as long as the prefactor is not negative (or else the variation wouldn't produce the required minimum / saddle point in the action). We choose the prefactor of the free particle to be written as $m/2$ to be consistent with our definition of the particle's momentum – otherwise we would simply be rescaling the mass of all particles by the same meaningless prefactor. Furthermore, a shift by an overall constant $\mathcal{L} \to \mathcal{L} + a$ does not affect any of these conclusions; we can conventionally take this overall constant to be zero.

4.3 Conservation laws and Noether's Theorem

Conserved charges What is the condition for a quantity to be conserved in some physical system? We mean that the value of that quantity stays constant – it doesn't change with time as the system evolves according to its equations of motion. Mathematically, the statement is as follows: a function $Q(q_a, \dot{q}_a, t)$ of the coordinates q_a, their time derivatives $\dot{q}_a = dq_a/dt$, and the time t is called a constant of motion, or *conserved quantity*, when Q remains constant along the path followed by the system. This is true if the total time derivative of Q vanishes whenever the $q_a(t)$ satisfies Lagrange's equations,

$$\frac{dQ}{dt} = 0 \quad \Leftrightarrow \quad Q \text{ is conserved}$$

Explicitly using the chain rule, this implies the following property of Q:

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial q}\dot{q} + \frac{\partial Q}{\partial \dot{q}}\ddot{q} + \frac{\partial Q}{\partial t} = 0$$

We will call Q satisfying these features a *conserved charge*.

A very common example is as follows. Suppose that $\mathcal{L}$ is independent of a particular coordinate q_a. Then, $\frac{\partial \mathcal{L}}{\partial \dot{q}_a}$ is a conserved quantity. Why? Because of the Euler-Lagrange equations:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_a} = \frac{\partial \mathcal{L}}{\partial q_a} = 0 \qquad \text{if } \mathcal{L} \text{ doesn't explicitly depend on } q_a$$

We call the quantity $\partial\mathcal{L}/\partial\dot{q}_a$ the *canonical / generalized momentum* associated to the coordinate q_a, which we call p_a:

$$\frac{\partial \mathcal{L}}{\partial \dot{q}_a} \equiv p_a \qquad \text{generalized momentum}$$

So the statement is that whenever the Lagrangian does not explicitly depend on one of the coordinates q_a, that coordinates generalized momentum p_a defines a conserved charge of the dynamics.

For instance, a particle moving in some potential $U(\vec{x})$ with kinetic energy $\frac{1}{2}m\dot{\vec{x}}^2$ has generalized momentum $\vec{p} = m\dot{\vec{x}}$, which is just the linear momentum of the particle. The Euler-Lagrange equations relate the time derivative of this momentum to the force exerted on the particle, which is the gradient of the potential,

$$\frac{d\vec{p}}{dt} = \vec{\nabla}U$$

If the potential does not depend explicitly on one of the coordinates x_i, then the right-hand-side is zero and its momentum $p_i = m\dot{x}_i$ is conserved.

Symmetries of the Lagrangian and Noether's theorem At the level of the Lagrangian, a symmetry is a transformation that only changes the Lagrangian up to a total derivative with respect to time:

$$\mathcal{L} \to \mathcal{L} + \frac{dF}{dt}$$

Why? Then the action then only changes by a constant boundary term,

$$S \to S + \int_{t_i}^{t_f} \frac{d}{dt}F = S + F(t_f) - F(t_i)$$

which cannot affect the equations of motion we derive from varying that action. (Recall that we are considering a path between fixed endpoints where the variation of the endpoints is fixed, so these extra boundary terms are just equal to zero.)

Mathematician Emmy Noether made a remarkable observation in 1918 which connects symmetries of the Lagrangian with conserved quantities. Her theorem can be stated as follows:

Noether's theorem: every continuous symmetry of the Lagrangian is associated to a conserved quantity.

Noether's theorem explains and in some cases predicts the existence of fundamental conserved quantities in nature. First we will prove this theorem in order to understand the general connection between conserved quantities and symmetries; then we will apply it in a number of examples. The proof will follow the following steps: (1) we will do a transformation that changes the coordinates by some infinitesimal amount which we can explicitly check is a symmetry of the Lagrangian. (2) We will calculate how the Lagrangian changes as a result of the transformation, using general principles and Noether's theorem. And finally (3) By relating the results of (1) and (2), we will identify a conserved quantity associated to the symmetry.

To start with part (1): consider a transformation of the generalized coordinates of the form,

$$q_a(t) \to q_a(t) + \delta q_a \tag{4.9}$$

where the change δq_a could in principle depend on the other coordinates. Importantly, we assume that this is a *continuous transformation*, in that we can make δq_i as small as we like until the transformation is infinitesimally small. (So in general, in our explicit examples below we will identify δq_a with some function $f_a(q_a)$ of the coordinates, times some real parameter ϵ that can be tuned to zero in order to consider the infinitesimal version of the transformation.)

This transformation is a symmetry of the Lagrangian and therefore won't affect the equations of motion, if it only changes the Lagrangian by a total derivative,

$$\delta\mathcal{L} \equiv \mathcal{L}(q_a + \delta q_a, \dot{q}_a + \delta\dot{q}_a, t + \delta t) - \mathcal{L}(q_a, \dot{q}_a, t) = \frac{dF}{dt} \;\Rightarrow\; \text{transformation is a symmetry of } \mathcal{L} \tag{4.10}$$

Of course, F might just be zero, so that the Lagrangian is completely invariant under the transformation (4.9).

(2) On the other hand, we can check the change in the Lagrangian $\delta\mathcal{L}$ due to this transformation via the chain rule:

$$\delta\mathcal{L} = \sum_i \left(\frac{\partial\mathcal{L}}{\partial q_a}\delta q_a + \frac{\partial\mathcal{L}}{\partial\dot{q}_a}\delta\dot{q}_a \right) + \frac{\partial\mathcal{L}}{\partial t}\delta t \tag{4.11}$$

The last term is only pertinent if the Lagrangian has an *explicit* dependence on time, and/or if the transformation explicitly involves the time coordinate so that $t \to t + \delta t$ – while this is not typical, later we will consider a special case of a transformation in time so we have included the term for completeness. The other terms can be rewritten in a nice way as follows. The second term is proportional to the generalized momentum $p_a = \partial\mathcal{L}/\partial\dot{q}_a$. In the first term we can use the Euler-Lagrange equations to rewrite

$$\frac{\partial\mathcal{L}}{\partial q_a} = \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_a} = \frac{dp_a}{dt}$$

So, this expression actually takes the form of almost a total derivative:

$$\delta\mathcal{L} = \sum_a \left(\dot{p}_a\delta q_a + p_a\delta\dot{q}_a\right) + \frac{\partial\mathcal{L}}{\partial t}\delta t = \frac{d}{dt}\sum_a p_a\delta q_a + \frac{\partial\mathcal{L}}{\partial t}\delta t \tag{4.12}$$

On to step (3). Suppose that either the Lagrangian does not explicitly depend on time, and/or the transformation does not explicitly involve the time coordinate, so that in either case the last term is zero. Comparing (4.11) with (4.10), we see that we can identify that the following total time derivative vanishes,

$$\frac{d}{dt}\left(\sum_a p_a\delta q_a - F\right) = 0\,.$$

In other words, the quantity in parenthesis is conserved! This quantity is the conserved charge associated to the symmetry transformation (4.9),

$$Q = \sum_a p_a \delta q_a - F \quad \text{is conserved} \quad \Leftrightarrow \quad \frac{dQ}{dt} = 0 \tag{4.13}$$

Example: Spatial translations & the homogeneity of space: As a first exploration of this theorem, let's consider the example we studied around (3.4): we have two masses, with a force acting between them directed from $\vec{x}_2 \to \vec{x}_1$. (This might correspond, for instance, to the gravitational force between the two masses, or the Coulomb force between two massive charged particles, or to some other general constant force that is a function of the distance $r = |\vec{x}_1 - \vec{x}_2|$). The Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec{x}}_1^2 + \frac{1}{2} m_2 \dot{\vec{x}}_2^2 - U(|\vec{x}_1 - \vec{x}_2|) \tag{4.14}$$

In a previous lecture we verified explicitly that Newton's equations for this system are invariant under the spatial translations of all vectors by the same constant amount,

$$\vec{x}_i \to \vec{x}_i + \vec{a}$$

for $\vec{a}$ some constant 3-vector, where here $i = 1, 2$ labels the two particles.

Accordingly, the Lagrangian for this system has a symmetry under spatial translations. To identify the conserved quantity in Noether's theorem, it is useful to rewrite the transformation in the way that we derived the theorem, as,

$$\vec{x}_i \to \vec{x}_i + \delta\vec{x}_i \,, \qquad \delta\vec{x}_i = \epsilon\vec{n} \tag{4.15}$$

where $\vec{n}$ is a constant vector pointed in an arbitrary direction, and ϵ is some real parameter that we can tune to zero. (In other words we are doing a redundant rewriting $\vec{a} = \epsilon\vec{n}$ for the convenience of identifying an explicit parameter ϵ that can be tuned to zero as we consider smaller and smaller translations.) We can explicitly check that the transformation (4.15) leaves the Lagrangian invariant,

$$\delta\mathcal{L} = \mathcal{L}(\vec{x}_i + \delta\vec{x}_i, \dot{\vec{x}}_i + \delta\dot{\vec{x}}_i) - \mathcal{L}(\vec{x}_i, \dot{\vec{x}}_i) = 0$$

This follows from the fact that $\dot{\vec{n}} = 0$ so that the kinetic terms don't change, and the constant shift by $\epsilon\vec{n}$ cancels out of the difference in the potential. Thus, spatial translations are a symmetry of the Lagrangian, with $F = 0$. (And of course, the transformation we are considering does not affect time so both $\partial\mathcal{L}/\partial t = 0$ and $\delta t = 0$ in (4.12).)

Turning the crank with Noether's theorem, this symmetry leads to the following conserved charge:

$$Q = \sum_{i=1}^{2} \vec{p}_i \cdot \delta\vec{q}_i = \epsilon\,(\vec{p}_1 + \vec{p}_2) \cdot \vec{n}$$

where $\vec{p}_1$ and $\vec{p}_2$ are the linear momenta of the particles, $\vec{p}_i = m_i\dot{\vec{x}}_i$, so that $\vec{p}_1 + \vec{p}_2$ is the total linear momentum of the 2-particle system. Thus, this charge is simply equal to the total linear momentum in the direction $\vec{n}$. Since this is a constant of motion for *any* arbitrary constant vector $\vec{n}$, we conclude that the total momentum $\vec{p}_1 + \vec{p}_2$ is conserved.

This result readily generalizes to a system of N particles whose potential depends only on the distances of the various particles to one another: the general statement is that the invariance

of the Lagrangian under spatial translations – which will hold true whenever the system is spatially homogeneous – implies conservation of the total linear momentum.

——— *End Lecture 16.*

Example: Rotations & the isotropy of space: Recall that we also have previously verified that the equations of motion for this 2-particle system are invariant under arbitrary coordinate rotations, where the $\vec{x}_i$ are rotated by the same amount. Such rotations are enacted by orthogonal transformations $\vec{x}_i \to O\vec{x}_i$ for O a matrix in the group $O(3)$. We can verify that the Lagrangian (4.14) is invariant under this transformation, so that the Lagrangian is symmetric under the group $O(3)$. This is straightforward to show: we have already verified that this transformation leaves the potential invariant, since[13]

$$|\vec{x}_1 - \vec{x}_2| \to |O(\vec{x}_1 - \vec{x}_2)| = |\vec{x}_1 - \vec{x}_2|\,.$$

It will also leave the kinetic terms invariant since

$$\dot{\vec{x}}_1^2 = (\dot{\vec{x}}_1)^T \dot{\vec{x}}_1 \quad \to \quad (O\dot{\vec{x}}_1)^T O\dot{\vec{x}}_1 = (\dot{\vec{x}}_1)^T O^T O\dot{\vec{x}}_1 = \dot{\vec{x}}_1^2$$

and similarly for the other particle. Therefore, $\delta\mathcal{L} = 0$ under the orthogonal transformation $\vec{x}_i \to O\vec{x}_i$ (so in particular, $F = 0$), and we can ask: what is the corresponding conserved quantity?

To apply Noether's theorem, we need to be able to consider rotations by matrices O that can be taken to be infinitesimally small, so that we can express the coordinate transformation as $\vec{x}_i \to \vec{x}_i + \delta\vec{x}_i$ for some δx_i that can be shrunk to zero if we like. Formally, this is the statement that the transformation can be continuously connected to the identity transformation, since we are looking for a matrix O where we can expand

$$O\vec{x}_i = (\mathbb{1} + \epsilon T + \dots)\,\vec{x}_i \quad \Rightarrow \quad \delta\vec{x}_i = \epsilon M \vec{x}_i \tag{4.16}$$

for some matrix T (and where we drop higher order terms in the expansion). So, we should focus on the continuous subgroup generated by rotations about some axis $\vec{n}$, since we can always perform a rotation by an infinitesimal amount. This is the subgroup $SO(3)$ of rotations in three dimensions. By contrast, a reflection about an axis can *not* be written in an infinitesimal version because reflections form discrete $\mathbb{Z}_2$ subgroup of $O(3)$ – they are binary operations and cannot be expressed in terms of a small parameter as in (4.16).

To be as explicit as possible, let's consider first a rotation about the $\vec{z}$-axis by some angle θ, which we know is enacted by the rotation matrix

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)$$

(*i.e.*, we consider the group element $O = R_z(\theta)$). The infinitesimal version of this transformation is given by expanding for small θ; we can take $\theta = \epsilon \ll 1$, and expand

$$R_z(\epsilon) \approx \begin{pmatrix} 1 & -\epsilon & 0 \\ \epsilon & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \epsilon \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.17}$$

[13] Recall this follows from $O^T O = \mathbb{1}$, since

$$|O(\vec{x}_1 - \vec{x}_2)| = \sqrt{(O(\vec{x}_1 - \vec{x}_2))^T\,(\vec{x}_1 - \vec{x}_2))} = \sqrt{(\vec{x}_1 - \vec{x}_2)^T O^T O(\vec{x}_1 - \vec{x}_2)}$$
$$= \sqrt{(\vec{x}_1 - \vec{x}_2)^T(\vec{x}_1 - \vec{x}_2)} = |\vec{x}_1 - \vec{x}_2|\,.$$

where we used $\cos\epsilon = 1 - \frac{\epsilon^2}{2} + \ldots$, and $\sin\epsilon = \epsilon + \ldots$ for small $\epsilon \ll 1$ (with the dots denoting terms of cubic and higher order in ϵ), and then kept only the leading order terms linear in ϵ. We therefore identify the matrix T in (4.16) with the matrix multiplying the ϵ above. Acting with $R_z(\epsilon)$ on the coordinates $\vec{x}_i$, this leads to the transformation

$$\begin{aligned} x_i &\to x_i - \epsilon y_i \\ y_i &\to y_i + \epsilon x_i \\ z_i &\to z_i \end{aligned}$$

which we recognize as the cross product of the unit $\hat{z}$-vector that we are rotating about, times the coordinate vector $\vec{x}_i = x_i\hat{x} + y_i\hat{y} + z_i\hat{z}$,

$$\vec{x}_i \;\to\; \vec{x}_i + \epsilon\,\hat{z}\times\vec{x}_i = \vec{x}_i + \epsilon\,(x_i\hat{y} - y_i\hat{x}) = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix} + \epsilon \begin{pmatrix} -y_i \\ x_i \\ 0 \end{pmatrix}$$

(At this point you might guess at how this formula will generalize for an infinitesimal rotation about an arbitrary axis $\hat{n}$, and you will explore this case in the Exercise below.) To summarize, we identify

$$\delta\vec{x}_i = \epsilon\,\hat{z}\times\vec{x}_i$$

in the infinitesimal version of the transformation.

Having identified the effect of the infinitesimal rotation on the coordinates, we can now apply Noether's theorem! The corresponding conserved charge is

$$Q = \sum_i \vec{p}_i\cdot\delta\vec{x}_i = \epsilon\sum_i \vec{p}_i\cdot(\hat{z}\times\vec{x}_i) = \epsilon\sum_i \hat{z}\cdot(\vec{x}_i\times\vec{p}_i)$$

where we pulled out the overall factor of ϵ, and used the identity $\vec{a}\cdot(\vec{b}\times\vec{c}) = b\cdot(\vec{c}\times\vec{a})$ to conveniently rewrite the cross product. We now recognize $\vec{x}_i\times\vec{p}_i$ as the angular momentum $\vec{L}_i$ of the i'th particle, so that the sum is over the z-components of the angular momenta:

$$Q = \epsilon\left[(\vec{L}_1)_z + (\vec{L}_2)_z\right] = \epsilon(\vec{L}_{\rm tot})_z$$

which is simply the total angular momentum of the system of particles about the z-axis. The generalization of this result to a rotation about an arbitrary axis leads to the statement: the invariance of the Lagrangian under rotations – which will hold true whenever the system is spatially isotropic – implies conservation of the total angular momentum.

Exercise 4.2

Suppose the Lagrangian of a system of particles in 3 dimensions is invariant under infinitesimal rotations about an arbitrary axis; *i.e.* that $\mathcal{L}$ is invariant under the transformation

$$\vec{x}_i \to \vec{x}_i + \epsilon\hat{n}\times\vec{x}_i$$

where i labels the different particles, and where $\hat{n}$ is an arbitrary unit vector.

(a) Name two physical systems that have this symmetry.

(b) Show that Noether's theorem implies the conservation of total angular momentum for this system.

Exercise 4.3

Consider a system of particles in 2 dimensions (x, y), whose Lagrangian is invariant under rotations of the particles' coordinates by R_θ given in (2.8).

(a) By Taylor expanding for infinitesimal angles $\theta \ll 1$, identify the transformation of the coordinates $\delta\vec{x}_i$ for this transformation.

(b) List an example of an element of $O(2)$ (*i.e.* write an explicit $O(2)$ matrix) that cannot be written as implementing an *infinitesimal* transformation of the coordinates (x, y).

Example: Time translations & the homogeneity of time: What is the symmetry associated with conservation of energy? The answer is time translation symmetry: the Lagrangian must be invariant under the explicit time translation symmetry

$$t \to t + \delta t \tag{4.18}$$

for δt some constant, so that $\partial\mathcal{L}/\partial t = 0$.

We can show this as follows. Suppose the Lagrangian is time translation invariant, so that $\partial\mathcal{L}/\partial t = 0$. For concreteness, let's consider the Lagrangian of a particle with the standard kinetic energy and potential energy $U(q)$:

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{q}}^2 - U(q)\,.$$

Step 1 is to compute what F is; *i.e.* to compute $\delta\mathcal{L}$ explicitly under the transformation (4.18). (Hint: unlike the previous examples, in this case F will not be 0!) To do this, we need to recognize that the *explicit* transformation of the time coordinate (4.18) leads to an *implicit* transformation of the coordinates q and $\dot{q}$. In particular, to be in line with our notation for spatial translations let's consider an infinitesimal transformation $\delta t = \epsilon n$ where n is some arbitrary constant, and ϵ is a parameter we can tune to 0 as we like. Taylor expanding, this leads to the following transformations of the coordinates and generalized momenta:

$$\begin{aligned}
&t \to t + \epsilon n \\
&\vec{q} \to \vec{q}(t+\delta t) = \vec{q}(t) + (\epsilon n)\dot{\vec{q}} + \dots \\
&\dot{\vec{q}} \to \dot{\vec{q}}(t+\delta t) = \dot{\vec{q}} + (\epsilon n)\ddot{\vec{q}} + \dots
\end{aligned}$$

The explicit transformation $t \to t + \delta t$ of the time coordinate thus leads to a change in $q(t)$ and $\dot{q}(t)$; in particular, we have identified

$$\delta\vec{q} = (\epsilon n)\dot{\vec{q}}, \qquad \delta\dot{\vec{q}} = (\epsilon n)\ddot{\vec{q}}. \tag{4.19}$$

We can compute the resulting change in the Lagrangian as follows:

$$\begin{aligned}
\delta\mathcal{L} &= \mathcal{L}(q+\delta q, \dot{q}+\delta\dot{q}, t+\delta t) - \mathcal{L}(q,\dot{q},t) \\
&= \frac{1}{2}m\left(\dot{\vec{q}} + \delta\dot{\vec{q}}\right)^2 - U(q+\delta q) - \mathcal{L}(q,\dot{q},t) \\
&= \frac{1}{2}m\dot{\vec{q}}^2 + m\dot{\vec{q}}\cdot\delta\dot{\vec{q}} - U(q) - \delta\vec{q}\cdot\vec{\nabla}U - \left[\frac{1}{2}m\dot{\vec{q}}^2 - U(q)\right] + \dots \\
&= m\dot{\vec{q}}\cdot\delta\dot{\vec{q}} - \delta\vec{q}\cdot\vec{\nabla}U \\
&= \epsilon n\left(m\dot{\vec{q}}\cdot\ddot{\vec{q}} - \dot{\vec{q}}\cdot\vec{\nabla}U\right)
\end{aligned} \tag{4.20}$$

where we expanded to linear order in ϵ, and then in the last line we substituted for (4.19). Now we can make a clever observation: the quantity in parenthesis is precisely equal to the total derivative $\frac{d\mathcal{L}}{dt}$. We can see this by computing

$$\begin{aligned}\frac{d\mathcal{L}}{dt} &= \frac{\partial \mathcal{L}}{\partial \vec{q}} \cdot \dot{\vec{q}} + \frac{\partial \mathcal{L}}{\partial \dot{\vec{q}}} \cdot \ddot{\vec{q}} + \frac{\partial \mathcal{L}}{\partial t} \\ &= -\dot{\vec{q}} \cdot \vec{\nabla} U + m\dot{\vec{q}} \cdot \ddot{\vec{q}}\end{aligned}$$

where we first just applied the chain rule, and then used that for this Lagrangian $\partial \mathcal{L}/\partial t = 0$ and substituted for $\partial \mathcal{L}/\partial \vec{q} = -\partial U/\partial \vec{q}$ and $\partial \mathcal{L}/\partial \dot{\vec{q}} = \vec{p} = m\dot{\vec{q}}$. We recognize this quantity as precisely the expression in the parenthesis in the last line of (4.20), so that in fact we have found that

$$\delta \mathcal{L} = \epsilon n \frac{d\mathcal{L}}{dt} \quad \Leftrightarrow \quad F = (\epsilon n)\mathcal{L}$$

which has allowed us to identify that under infinitesimal time translations, the Lagrangian transforms up to a total time derivative of F which is just proportional to $\mathcal{L}$ itself!

Then, step 2 is to apply Noether's theorem, and therefore identify the corresponding conserved charge from (4.13),

$$Q = \vec{p} \cdot \delta \vec{q} - F = (\epsilon n) \left(\vec{p} \cdot \dot{\vec{q}} - \mathcal{L} \right)$$

The expression in parenthesis is by definition what we call the *Hamiltonian* of the system, usually denoted by the letter H, which in the systems we're considering is simply the total energy. This result says that when the Lagrangian is invariant up to a total derivative under time translations, so that it does not explicitly depend on time – which will hold true when the system is homogeneous in time – the Hamiltonian defined by

$$H = \vec{p} \cdot \dot{\vec{q}} - \mathcal{L} \tag{4.21}$$

is a conserved quantity, so that energy is conserved.

At this point, we have covered the consequences of the symmetries of spatial translations, rotations, and time translations in 3 dimensions, which recall form the Euclidean group. Thus, we can succinctly summarize the results of this subsection as the statement that: when a Lagrangian describing some physical system is invariant under the Euclidean group, that system conserves energy, linear momentum, and angular momentum. This answers the *why* as to why certain physical systems satisfy these conservation laws.

Exercise 4.4

The Lagrangian for a particle moving in a uniform force field $\vec{F}$ is given by,

$$\mathcal{L} = \frac{1}{2} m \dot{\vec{x}}^2 + \vec{F} \cdot \vec{x} \, .$$

Consider specifically a ball of mass m in free fall in the Earth's uniform gravitational field, which close to the Earth has this Lagrangian with $\vec{F} = -mg\hat{z}$.

(a) Write the Euler-Lagrange equations that the ball's motion satisfies.

(b) Show that the components of linear momentum perpendicular to $\vec{F}$ are preserved. What symmetry of the Lagrangian is this related to?

(c) Show that the component of angular momentum along $\vec{F}$ is preserved. What symmetry of the Lagrangian is this related to?

(d) Show that the Hamiltonian $H = \vec{p}\cdot\dot{\vec{q}} - \mathcal{L}$ of this system is equal to the total energy. Is energy conserved?

Worked example: the 2-body problem Let's put our conservation laws to use in solving a classic problem, the 2-body problem. The two-body problem consists of two objects interacting through a central force. You might consider, for example, the gravitational interaction between the sun (body 1) and Pluto (body 2). In terms of the center of mass coordinate $\vec{R}$ and separation $\vec{r}_{12} = \vec{x}_1 - \vec{x}_2$ between the bodies, the Lagrangian for this system can be written as

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}}^2 + \frac{1}{2}\mu\dot{\vec{r}}_{12}^2 + \frac{Gm_1m_2}{|\vec{r}_{12}|}$$

where the reduced mass μ is

$$\mu = \frac{m_1m_2}{m_1 + m_2}$$

The Lagrangian completely decouples into a piece describing the constant motion of the center of mass $\vec{R}$, and a piece describing the separation between the two masses. Note that since the mass of the sun is so much larger than the mass of Pluto ($m_2 = 1.3\times10^{22}\,kg$ and $m_1 = 2.0\times10^{30}\,kg$), the reduced mass is approximately just Pluto's mass,

$$\mu \approx m_2\,, \qquad m_1 + m_2 \approx m_1$$

Since the center of mass motion only involves the kinetic term, its linear momentum is conserved – a fact we can use to separately solve for the motion of the center of mass of the 2-body system:

$$(m_1 + m_2)\dot{\vec{R}} = \vec{p}_{\rm com} = \text{constant} \quad \Rightarrow \quad \vec{R}(t) = \frac{\vec{p}_{\rm com}}{(m_1 + m_2)}t + \vec{R}(0)$$

The center of mass just evolves linearly in time.

——— *End Lecture 17.*

Next let's focus on the separation piece in order to understand Pluto's orbits around the sun. We know from Noether's theorem that the angular momentum $\vec{L} = \vec{r}_{12}\times\vec{p}_{12}$ is conserved, where $\vec{p}_{12}$ is the momentum conjugate to $\vec{r}_{12}$, $\vec{p}_{12} = \mu\dot{\vec{r}}_{12}$. Since $\vec{L}$ is perpendicular to $\vec{r}_{12}$, the motion of the orbit must lie in a plane perpendicular to $\vec{L}$. Therefore, it would be smart to choose our coordinates so that (r,ϕ) are polar coordinates in that plane. Calling the plane in which the particle travels in the x-y plane, so that the angular momentum is oriented along the z-axis, we can use polar coordinates

$$(r_{12})_x = r\cos\phi\,, \qquad (r_{12})_y = r\sin\phi$$

and rewrite the Lagrangian for the separation piece as,

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{Gm_1m_2}{r} \tag{4.22}$$

Exercise 4.5

(a) Show that $J = \mu r^2 \dot{\phi}$ is a conserved quantity. (Hint: the potential of (4.22) does not depend on ϕ.) J corresponds to the magnitude of the angular momentum $\vec{L}$.

(b) Verify that the transformation $\phi \to \phi + c$, $r \to r$ of $\mathcal{L}$ leads to the conserved charge J by application of Noether's theorem. Therefore, this conserved charge is due to the rotational symmetry of the Lagrangian.

(c) Show that the energy $E = \frac{1}{2}\mu \dot{r}^2 + U_{\text{eff}}(r)$ is conserved, where we have defined the effective potential

$$U_{\text{eff}}(r) = -\frac{Gm_1 m_2}{r} + \frac{J^2}{2\mu r^2}\,.$$

At this point we have reduced our problem to motion in the plane with $J = \mu r^2 \dot{\phi}$ a constant of motion. At this point it is useful to define the following combination as an *effective potential*,

$$U_{\text{eff}}(r) = -\frac{Gm_1 m_2}{r} + \frac{J^2}{2\mu r^2}$$

It is a straightforward exercise to then show that the the Euler Lagrange equations of this system reduce simply to

$$\mu \ddot{r} = -\frac{\partial}{\partial r} U_{\text{eff}}(r) \tag{4.23}$$

How to solve this? Actually, we don't need to solve (4.23) at all! As you will show in Exercise 4.5, conservation of energy leads to the constant of motion

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{\text{eff}}(r) \tag{4.24}$$

and we can put our two constants of motion to use to solve for the orbits.

Let us summarize our progress: we have reduced the 2-body problem to the problem of a body of reduced mass μ moving in the effective potential $U_{\text{eff}}(r)$, and have identified two constants of motion for the system: E (energy) and J (magnitude of angular momentum perpendicular to the plane of motion). These two constants of motion will allow us to solve for the orbit: using conservation of energy (4.24), we can determine $r(t)$ by integrating up from $t = 0$,

$$E = \frac{1}{2}\mu \dot{r}^2 + U_{\text{eff}}(r) \quad \Rightarrow \quad \sqrt{\frac{\mu}{2}} \int_{r(0)}^{r(t)} \frac{dr}{\sqrt{E - U_{\text{eff}}(r)}} = t$$

Then, we can substitute $r(t)$ into the equation for J to solve

$$J = \mu r^2 \dot{\phi} \quad \Rightarrow \quad \phi(t) - \phi(0) = \int_0^t \frac{J}{\mu} \frac{dt}{r(t)^2}$$

In summary, conservation of center of mass momentum allows us to determine the time evolution of the center of mass, and then conservation of energy and angular momentum allows us to solve for the motion of the small body around the big body. We need to specify initial conditions $\phi(0)$, $r(0)$, and $\vec{R}(0)$, as well as the constant values of E and J for the orbit.

We won't actually solve these equations here (this is an excellent problem for an advanced course in classical mechanics or a numerical methods course, but not so much for this course..),

but the summary is: at the minimum of the effective potential there is a solution with a perfectly circular orbit. Otherwise, when $E < 0$ the orbits are elliptic, and when $E > 0$ the orbits are hyperbolic. If you are interested in a more detailed analysis of the solutions with plots, I recommend taking a look at these MIT OCW lecture notes – the different types of orbits are discussed in sections 25.4-5.

Math Aside: generating Lie groups An important assumption of Noether's theorem was that we could expand our symmetry action on the coordinates as in (4.9),

$$q_a \;\to\; q'_a = q_a + \delta q_a\,, \qquad \delta q_a = \epsilon f_a(q)$$

for some function $f_a(q)$ of the coordinates that will depend on the particular symmetry, and ϵ a parameter that can be tuned to zero. If the symmetry under consideration is a matrix group enacted by matrices M, then this is the statement that we can expand (using a vector notation)

$$q' = Mq = (\mathbb{1} + \epsilon T + \dots)\, q \quad \Rightarrow \quad \delta q = \epsilon T q$$

where T is a matrix with the same dimensions as M. This was the case in the example of angular momentum conservation, where the symmetry group under consideration was the rotation group in three dimensions, $SO(3)$, so that we were able to identify δq from (4.16) for rotations about the z-axis. You considered a similar example in Exercise 4.3 for the group $SO(2)$ of rotations in 2-dimensions, where you showed that the matrix T in that case is

$$SO(2): \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tag{4.25}$$

When the matrix group is a Lie group (as is the case for $SO(2)$, $SO(3)$, *etc.*), it turns out that the matrices T actually *generate the entire group*: they are identified with the generators of the Lie group. This is basically because ϵT is the first term in the Taylor expansion of the matrix exponential,

$$M(\epsilon) = e^{\epsilon T} = \mathbb{1} + \epsilon T + \frac{1}{2!}(\epsilon T)^2 + \dots \tag{4.26}$$

so that given the generator T, we can just exponentiate it (via the matrix exponential) to completely determine the group element M labeled by parameter ϵ. Knowledge of T is as good as knowledge of M, since they are related by (4.26).

There is a generator for each independent parameter of the group. For $SO(2)$, T given in (4.25) is the only generator – $SO(2)$ is an abelian group that only depends on one real parameter θ, and a generic group element can be written as $R_\theta = e^{\theta T}$ for T the matrix in (4.25). Another example is $SO(3)$, which we've learned is a nonabelian group that depends on three real parameters that parameterize rotations about each of the three axes. In (4.17) we showed that the generator of rotations about the z-axis is the matrix multiplying the ϵ there, call it T_3. Performing the similar exercise for rotations about the other two axes by also expanding $R_x(\epsilon)$ and $R_y(\epsilon)$ for small ϵ allows us to identify the other two generators, with result:[14]

$$SO(3): \qquad T_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{4.27}$$

[14] Note that the generators defined in this way are fixed up to a constant scalar prefactor. Conventionally that prefactor is taken to be $-i$, so if you look up these relations on Wikipedia you might see a bunch of i's floating around.

Then, a generic element of $SO(3)$ that describes a rotation about an axis $\vec{n} = \alpha\hat{x} + \beta\hat{y} + \gamma\hat{z}$ can be written as $R(\alpha, \beta, \gamma) = e^{\alpha T_1 + \beta T_2 + \gamma T_3}$.

Of course, in these lectures we've arrived at the generators T by expanding the group elements. Somewhat more abstractly, we can go the other way around: the entire Lie group is defined by the generators T, and in particular how they commute with each other. You can verify that the matrices (4.27) satisfy the following relations:

$$T_1T_2 - T_2T_1 = T_3\,, \qquad T_2T_3 - T_3T_2 = T_1\,, \qquad T_3T_1 - T_1T_3 = T_2\,,$$

which can be succinctly written in terms of the commutator $[A, B] = AB - BA$ as,

$$[T_i, T_j] = \sum_{k=1,2,3} \epsilon_{ijk} T_k\,, \qquad i, j, k = 1, 2, 3 \tag{4.28}$$

where the indices i, j, k each run over the three generators, $1, 2, 3$. Here the Levi-Civita symbol ϵ_{ijk} is defined as $+1$ when its indices are a cyclic (even) permutation of $(1, 2, 3)$, -1 when its indices are an anticyclic (odd) permutation of $(1, 2, 3)$, and 0 when two or more indices coincide:

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3), (3,1,2), (2,3,1) \\ -1 & \text{if } (i,j,k) = (1,3,2), (3,2,1), (2,1,3) \\ 0 & \text{if } i = j \text{ or } j = k \text{ or } k = i \end{cases}$$

For example: we can evaluate (4.28) for a couple of values of i, j as follows:

$$[T_1, T_1] = \epsilon_{111}T_1 + \epsilon_{112}T_2 + \epsilon_{113}T_3 = 0 + 0 + 0 = 0$$
$$[T_1, T_3] = \epsilon_{131}T_1 + \epsilon_{132}T_2 + \epsilon_{133}T_3 = 0 + (-1)T_2 + 0 = -T_2$$

The first equation simply reproduces the fact that a matrix always commutes with itself: $AA - AA = 0$. The second yields $[T_1, T_3] = -T_2$, or in other words $[T_3, T_1] = +T_2$ (since $[A, B] = -[B, A]$), which you can check is explicitly satisfied by the matrices (4.27). And so on.

The commutation relations (4.28) completely characterize the rotation group in three dimensions $SO(3)$, and can be taken as the fundamental *definition* of the group. Starting only from (4.28), one can determine a matrix representation of the generators T_1, T_2, T_3 by finding matrices that satisfy those commutation relations, and thereby by exponentiation determine all possible group elements $R(\alpha, \beta, \gamma)$. Different Lie groups will have different numbers of generators, and satisfy different commutation relations similar to (4.28) but with different constants than ϵ_{ijk} (known generally as *structure constants*) on the right-hand-side.[15]

For the purposes of identifying conserved charges via Noether's theorem, this discussion is useful because it allows us to make a general statement about the form of the conserved charge when the symmetry transformation is a Lie group: whenever a Lagrangian has a symmetry which forms a Lie group, there is one conserved charge Q_i for each generator of the Lie group T_i, given by

$$Q_i = \epsilon \sum_a p_a (T_i\, q)_a - F \tag{4.29}$$

where F is zero if $\mathcal{L}$ is completely invariant under the transformation.

[15] The general structure is $[T_i, T_j] = \sum_k f_{ijk} T_k$ where the indices i, j, k run over the number of generators, and where f_{ijk} are a collection of numbers known as structure constants that are different for different groups.

Exercise 4.6

Suppose a system with Lagrangian $\mathcal{L} = \frac{1}{2}m\dot{\vec{x}}^2 - U(\vec{x})$ is invariant under rotations about the x, y, z directions, and so has a symmetry group $SO(3)$ generated by the matrices T_1, T_2, T_3 in (4.27). Using (4.29), compute the three charges $Q_{1,2,3}$.

——— *End Lecture 18.*

4.4 The Poisson bracket and constants of motion

In this section we'll discuss a formal description of classical dynamics which makes it look almost identical to quantum mechanics – it will allow us to make the leap in the next section from conservation laws in classical systems to conservation laws in quantum systems. It will also provide a complementary and useful perspective on symmetries of classical systems.

From Lagrangian to Hamiltonian mechanics We've discussed how to build the Hamiltonian from the Lagrangian:

$$H = \sum_a p_a \dot{q}_a - \mathcal{L}$$

The Lagrangian is a function of q and $\dot{q}$ (and in principle t); this relation is the Legendre transform which takes us from the function $\mathcal{L}(q, \dot{q})$ to the function $H(q, p)$ where $p = \partial\mathcal{L}/\partial\dot{q}$. Given H, it is always possible to do the inverse Legendre transform back from H to $\mathcal{L}$:

$$\mathcal{L} = \sum_a \frac{\partial H}{\partial p_a} p_a - H$$

So, the two functions $\mathcal{L}(q, \dot{q})$ and $H(p, q)$ contain the same information, and it's really our choice as to which is more convenient to use. In the Hamiltonian formalism, q_a and p_a are the independent variables that label my system.

Passing from the Lagrangian to the Hamiltonian $H(p, q)$ formalism, the equations of motion can be shown to be given by Hamilton's equations:

$$\dot{p}_a = -\frac{\partial H}{\partial q_a}\,, \qquad \dot{q}_a = \frac{\partial H}{\partial p_a}\,.$$

There are various ways to derive these equations which we will not review here: one can obtain them directly from the definition of the Legendre transform, or also by varying the action with $\mathcal{L} \to H$ much as we did in the Lagrangian case. These equations can be nice to work with because we've swapped a set of 2nd order differential equations (the Euler-Lagrange equations) for twice the number of 1st order differential equations.

The Poisson bracket Let us make a definition which will prove itself useful to our study of symmetries in Hamiltonian systems. Let $f(q, p)$ and $g(q, p)$ be two functions of the coordinates q_a and generalized momenta p_a. The *Poisson bracket* between f and g is defined as

$$\{f, g\} = \sum_a \left(\frac{\partial f}{\partial q_a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q_a} \right)$$

This is a bit of a funny definition, so let's explore some of its properties.

- The Poisson bracket is antisymmetric under exchange of its arguments: $\{f, g\} = -\{g, f\}$.
- It is linear: $\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}$ for constants α, β.
- It satisfies what is known as the Leibniz rule: $\{fg, h\} = f\{g, h\} + \{f, h\}g$. This identity can be verified using from the Chain rule for partial derivatives.
- It satisfies the Jacobi identify: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$. This is not hard to show, it just takes some patience writing out all the terms and canceling them!
- Computing the Poisson bracket of the coordinates q and p yields the following result:

$$\{q_a, p_b\} = \delta_{ab}\,, \qquad \{q_a, q_b\} = 0\,, \qquad \{p_a, p_b\} = 0\,.$$

  This is because the q, p coordinates are independent variables, so for instance $\partial q_a/\partial p_b = 0$ and $\partial p_a/\partial q_b = 0$. For example, the first identity can be verified as follows:

$$\{q_a, p_b\} = \sum_c \left(\frac{\partial q_a}{\partial q_c}\frac{\partial p_b}{\partial p_c} - \frac{\partial q_a}{\partial p_c}\frac{\partial p_b}{\partial q_c}\right) = \sum_c (\delta_{ac}\delta_{bc} - 0) = \delta_{ab}$$

One reason this notation is useful is because we can express the equations of motion as Poisson bracket relations; one can verify by explicit computation that

$$\begin{aligned} \{H, q_a\} = \frac{\partial H}{\partial p_a}\,, \qquad \{H, p_a\} &= -\frac{\partial H}{\partial q_a} \\ \Rightarrow \dot{q}_a = \{H, q_a\}\,, \qquad \dot{p}_a &= \{H, p_a\} \end{aligned}$$

Aside: recall in Lagrangian mechanics we made a big deal of the fact that we could change variables $x \to q$ and keep the Euler-Lagrange equations the same. The analogous statement in this Hamiltonian language is that as long as you do a coordinate transformation $q_a \to \hat{q}_i, p_a \to \hat{p}_i$ which preserves the Poisson bracket structure,

$$\{\hat{q}_i, \hat{p}_j\} = \delta_{ij}\,, \qquad \{\hat{q}_i, \hat{q}_j\} = \{\hat{p}_i, \hat{p}_j\} = 0$$

then Hamilton's equations and the Poisson bracket are invariant under the transformation. Such coordinate transformations are called *canonical transformations*.

Symmetries in Hamiltonian systems How does the relationship between symmetries and conserved quantities manifest in the Hamiltonian language?

Our first claim is that if $\frac{\partial H}{\partial t} = 0$ so that H does not explicitly depend on time, then H is a constant of motion. This follows from the fact that

$$\frac{\partial H}{\partial t} = -\frac{\partial \mathcal{L}}{\partial t}\,. \tag{4.30}$$

Since we've already shown that $\frac{\partial \mathcal{L}}{\partial t} = 0$ implies that H is the conserved Noether's charge, (4.30) equivalently implies that H is a constant of motion. (4.30) is not hard to show, but we won't go through the details here.

What about more general constants of motion? We will need the following identity: for some function $f(q, p)$ of the variables q_a, p_a,

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,. \tag{4.31}$$

This identity can be shown as follows: with the chain rule we expand

$$\begin{aligned}\frac{df}{dt} &= \frac{\partial f}{\partial t} + \sum_a \left(\frac{\partial f}{\partial q_a}\dot{q}_a + \frac{\partial f}{\partial p_a}\dot{p}_a \right) \\ &= \frac{\partial f}{\partial t} + \sum_a \left(\frac{\partial f}{\partial q_a}\frac{\partial H}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial H}{\partial q_a} \right)\end{aligned}$$

where we used Hamilton's equations in the second line. We recognize the quantity in parenthesis as precisely the definition of $\{f, H\}$, which completes the proof. Therefore, if we can find a function $I(p, q)$ which is independent of t whose Poisson bracket with the Hamiltonian vanishes, I is a constant of motion:

$$\frac{dI}{dt} = 0 \quad \Rightarrow \quad \{I, H\} = 0$$

In other words, **a constants of motion Poisson commute with the Hamiltonian**.

So, the logic is as follows. Suppose we identify a Noether's charge Q, that naturally satisfies $dQ/dt = 0$ and defines a constant of motion. In the Hamiltonian formalism, this charge necessarily Poisson commutes with the Hamiltonian, $\{Q, H\} = 0$. On the other hand, suppose given some Hamiltonian we find some function I which satisfies $\{I, H\} = 0$, so that I is a constant of motion. This I can be related to the Noether's charge for some symmetry of the Lagrangian. The conserved charges can be mapped 1-to-1 onto each other, but there is some inherent ambiguity in relating Q and I; for instance, if I and J are both constants of motion satisfying $\{I, H\} = 0$ and $\{J, H\} = 0$, then (1) due to the Jacobi identity $\{I, J\}$ is *also* a constant of motion, and furthermore (2) multiplying either I or J by a scalar is also a constant of motion.

Symmetries beget symmetries This last point (1) is useful in itself! For example, suppose I know that $L_x = (\vec{x} \times \vec{p})_x = yp_z - zp_y$ and $L_y = (\vec{x} \times \vec{p})_y = zp_x - xp_z$ are conserved, *i.e.* both satisfy $\{L_x, H\} = 0$ and $\{L_y, H\} = 0$. We can compute their Poisson bracket as,

$$\begin{aligned}\{L_x, L_y\} &= \{yp_z - zp_y, zp_x - xp_z\} = \{yp_z, zp_x\} - \{yp_z, xp_z\} - \{zp_y, zp_x\} + \{zp_y, xp_z\} \\ &= yp_x\{p_z, z\} + p_y x\{z, p_z\} = xp_y - yp_x = L_z\end{aligned} \tag{4.32}$$

So: $\{L_x, L_y\} = L_z$. If L_x and L_y are conserved so that $\{L_x, H\} = \{L_y, H\} = 0$, then this automatically implies $\{L_z, H\} = 0$; by the Jacobi identity

$$\begin{aligned}\{L_z, H\} &= \{\{L_x, L_y\}, H\} = -\{H, \{L_x, L_y\}\} \\ &= \{L_y, \{H, L_x\}\} + \{L_x, \{L_y, H\}\} = 0\end{aligned}$$

So if L_x and L_y are conserved, L_z must *also* be conserved, and the total angular momentum vector $\vec{L}$ is conserved. *This is a general statement about any classical system.* Nice![16]

Exercise 4.7

Consider the example from Exercise 4.4 of a ball moving in the Earth's uniform gravitational field, with Lagrangian

$$\mathcal{L} = \frac{1}{2}m\dot{\vec{x}}^2 - mgz\,.$$

(a) Write the Hamiltonian for this system in terms of the variables $\vec{p} = (p_x, p_y, p_z)$ and

[16] Of course, the converse is not true: $\{L_z, H\} = 0$ does *not* imply that $\{L_x, H\} = \{L_y, H\} = 0$; it could be the case that $\{L_y, \{H, L_x\}\} = -\{L_x, \{L_y, H\}\}$.

$\vec{x} = (x, y, z)$.

(b) Compute the Poisson brackets $\{p_x, H\}$, $\{p_y, H\}$, $\{p_z, H\}$, and $\{L_z, H\}$. What can you conclude?

4.5 Lagrangians beyond classical mechanics

As we emphasized at the beginning of this section, part of the reason it is so useful to study Lagrangian mechanics is because many systems can be understood from variational principles. If we know the underlying symmetries of the system we're trying to describe, we can start from (1) the point of view that the Lagrangian should be invariant (up to a total derivative) under those symmetries, and then (2) use information about what that Lagrangian should reduce to in various limits, or what equations of motion we expect to get out in various limits, in order to make sure we have the right answer.

Relativistic systems form a good set of examples. We learned in the section on relativity that we can build Lorentz-invariant objects by summing over spacetime indices, with the correct insertions of the Minkowski metric η. If I wish to describe the Lagrangian for a system which is invariant under the Lorentz group, then a good starting point is to write down the possible Lorentz-invariant objects that have the right units.

For example, we've learned that the action for a non-relativistic particle is given by $S = \int dt(K - U)$. What is the action for a free relativistic point particle? We claim that the answer is as follows,

$$S = -mc \int \sqrt{-ds^2}\,, \qquad ds^2 = -(c\,dt)^2 + dx^2 + dy^2 + dz^2 \tag{4.33}$$

where the integral is over the spacetime trajectory (worldline) of the particle between fixed end points $(ct_i, \vec{x}_i)$ and (ct_f, x_f).

——— *End Lecture 19.*

There are a few equivalent ways to express this action; in particular, expanding out the integrand yields

$$\begin{aligned}\sqrt{-ds^2} &= \sqrt{(ct)^2 - dx^2 - dy^2 - dz^2} = c\,dt\sqrt{1 - \frac{1}{c^2}\left[\left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right]} \\ &= c\,dt\sqrt{1 - \frac{v^2}{c^2}} = cd\tau\end{aligned}$$

where v is the velocity of the particle, and recall that τ is the proper time elapsed in the particle's rest frame. So, we can rewrite the Lagrangian as minus the rest energy divided by the relativistic γ-factor,

$$\mathcal{L} = -mc^2\sqrt{1 - \frac{v^2}{c^2}}\,, \qquad S = \int_{t_i}^{t_f} \mathcal{L}\,dt \tag{4.34}$$

or equivalently, we can write that $S = -mc^2 \int_{\tau_i}^{\tau_f} d\tau$.

There are a number of checks that might convince you this is the right answer:

- The action is clearly invariant under the Lorentz group, since the invariant interval ds is invariant under this group. All observers will agree on the value of the action, regardless of which relative frame they are in.

  In fact, the action is just a constant times the integral over the invariant spacetime interval (up to a funny minus sign that we need due to the minus sign in the Minkowski metric) – in particular, compare this expression to (3.8). Varying the action leads to the equations of motion for the particle. This means that the particle follows a path through Minkowski space that minimizes the spacetime interval ds, or in other words, the relativistic particles follow *geodesics* in spacetime.

- At low speeds $v \ll c$, this Lagrangian reproduces our usual non-relativistic free particle Lagrangian. We can show this explicitly by expanding,

$$\mathcal{L} = -mc^2\sqrt{1-\frac{v^2}{c^2}} = -mc^2\left(1-\frac{1}{2}\frac{v^2}{c^2}+\dots\right) = \frac{1}{2}mv^2 - mc^2$$

  which is equal to the usual non-relativistic kinetic energy up to a constant (the rest energy of the particle).

- The canonical momentum is precisely identified as the relativistic momentum of the particle moving with velocity $\vec{v}$,

$$\vec{p} = \frac{\partial \mathcal{L}}{\partial \dot{\vec{x}}} = -\frac{1}{2}mc^2\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\left(-2\frac{\vec{v}}{c^2}\right) = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma_v m\vec{v}$$

  Standard application of the Euler-Lagrange equations yields $\dot{\vec{p}} = 0$.

- The Hamiltonian computed from this Lagrangian is precisely the relativistic energy of the particle with velocity v, $E = \gamma_v mc^2$:

$$\begin{aligned} H = \vec{p}\cdot\dot{\vec{x}} - \mathcal{L} &= \frac{mv^2}{\sqrt{1-\frac{v^2}{c^2}}} - (-mc^2)\sqrt{1-\frac{v^2}{c^2}} = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}\left(v^2+c^2-v^2\right) \\ &= \gamma_v mc^2 \end{aligned}$$

  Then, energy and momentum conservation for the relativistic free particle follow from direct application of Noether's theorem, since the Lagrangian $\mathcal{L}$ is independent of both t (so that H is a constant) and $\vec{x}$ (so that $\vec{p}$ is a constant).

Another nice example is the action for electromagnetism. As we've already noted, electromagnetism is naturally covariant under Lorentz transformations, and so we expect to be able to derive Maxwell's equations via variation of a relativistically invariant action. This action is,[17]

$$S = \int dt \int d^3x \left(-\frac{1}{4\mu_0}\mathrm{Tr}\left[(\eta G)^2\right] - J\cdot A\right) \tag{4.35}$$

[17] It is straightforward to verify that this action is invariant under Lorentz transformations: the first term transforms as,

$$\mathrm{Tr}(\eta G)^2 = \mathrm{Tr}\eta G\eta G \;\rightarrow\; \mathrm{Tr}\eta(\Lambda G\Lambda^T)\eta(\Lambda G\Lambda^T) = \mathrm{Tr}\eta\Lambda G\eta G\Lambda^T = \mathrm{Tr}\Lambda^T\eta\Lambda G\eta G = \mathrm{Tr}\eta G\eta G$$

where we used that $G \rightarrow \Lambda G\Lambda^T$ under a Lorentz transformation; the fact that $\Lambda^T\eta\Lambda = \eta$, and the property of the trace that $\mathrm{Tr}ABC = \mathrm{Tr}CAB$. The second term $J\cdot A$ is the dot product of two 4-vectors, and so is also invariant.

where $G_{\mu\nu}$ is the electromagnetic field strength tensor that we encountered in the previous section,

$$G_{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix},$$

$J^\mu = (c\rho, \vec{j})$ is the relativistic current density 4-vector, and $A^\mu = (\frac{\phi}{c}, \vec{A})$ is the vector potential, in terms of which the electric and magnetic fields can be expressed as

$$\vec{B} = \vec{\nabla} \times \vec{A}, \qquad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}.$$

Notice that the Lagrangian in this case is actually a Lagrangian *density*, since $S = \int dt\, \mathcal{L}$ where $\mathcal{L}$ itself involves an integral over space. This is because the basic objects in this Lagrangian are fields – the electric and magnetic fields, and vector potential – rather than coordinates.

Maxwell's equations are derived by varying this action, where we express the variation of the action in terms of the components of the vector potential A^μ. This takes a bit of work, since we need to rewrite everything in terms of the vector potential and then apply the variational principle. Explicitly, we can expand out

$$S = \int dt \int d^3x \left(\frac{1}{2\mu_0} \left(B^2 - \frac{1}{c^2} E^2 \right) - \vec{j} \cdot \vec{A} + \frac{1}{c} \rho\phi \right)$$

and then further expand $\vec{B}$ and $\vec{E}$ in terms of the components of A^μ. After some vector calculus identities and integrating by parts, you will find that varying with respect to A^μ leads to the two Maxwell's equations that follow from $\partial \cdot G = \mu_0 J^T$ (see Exercise 3.11).

As a final comment, we note that conservation of electric charge – which recall from (3.29) can be expressed as $\partial \cdot J = 0$ – follows by application of Noether's theorem, where the pertinent symmetry transformation is a *gauge transformation* of the vector potential $A^\mu \to A^\mu + \delta A^\mu$. Showing this explicitly is beyond the scope of these lectures, since it requires deriving the analogues of the Euler-Lagrange equations and Noether's theorem for the action (4.35) (which depends on fields..), and also requires a more in depth discussion of gauge transformations in electromagnetism.[18]

[18] For those interested in learning more, an interesting "intuitive" discussion of gauge transformations is given at this link.

5 Symmetry in Quantum Mechanics

Possibly helpful resources:

- **MOST RECOMMENDED:** *A First Course on Symmetry, Special Relativity, and Quantum Mechanics* by Kunstatter and Das, Section 10.8-9 on linear operators and symmetry in quantum mechanics.
- Supplemental reading: Feynman lectures on symmetry and conservation laws in quantum mechanics at this link.
- If you are familiar with bra-ket notation, Chapter 4.1-4.2 of the textbook *Modern Quantum Mechanics* by Sakurai-Napolitano has a nice discussion of symmetries in quantum mechanics, including parity.
- (Very) supplemental reading: University of Cambridge Professor David Tong's lecture notes on extending classical mechanics to quantum mechanics are a nice additional resource, see Section 4.8 at this link. Although we will not cover it in class, he gives a nice explanation for how the variational principle we learned in classical mechanics generalizes to quantum systems (section 4.8.1).
- A very nice but somewhat more advanced review of symmetries in quantum mechanics and particle physics by Professor Rischke in Germany can be found here.

5.1 Review: Wavefunctions, observables, and the Schrödinger equation

The wavefunction In quantum mechanics, the state of a system is described by the complex valued *wavefunction* $\Psi(\vec{x}, t)$, that describes everything that can be known about the system. The wavefunction is a complex function of space and time that is the ultimate DNA of the particle/system.

The wavefunction allows us to predict the statistical outcomes of measurements. As quantum mechanics is inherently probabilistic, we do not specify with certainty the location of a particle(s), but instead assign a probability for finding it some small region of space at a particular time: this probability density is determined by the wavefunction as

$$P(x,t) = \Psi(x,t)^* \Psi(x,t)\, dx = |\Psi(x,t)|^2\, dx$$

(Here and for much of this review we'll focus on formulas in 1 spatial dimension, and explain the generalization to 3 dimensions when needed.)

Observables All properties of the system that you can measure – momentum, kinetic energy, potential energy, color, *etc.* – are called *observables*. Observables O in quantum mechanics are represented by linear *operators* $\hat{O}$ (denoted with a hat): an object that acts on the wavefunction and gives another wavefunction as a result. The two most important operators you learn about in PHYS 200 are the position operator $\hat{x}$ and momentum operator $\hat{p}$. The position operator just acts via normal multiplication, so that we can replace $\hat{x} = x$ (the hat doesn't do anything special):

$$\hat{x} = x \quad \Rightarrow \quad \hat{x}\Psi(x,t) = x\Psi(x,t)$$

The momentum operator $\hat{p}$ is a derivative operator:

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \quad \Rightarrow \hat{p}\Psi(x,t) = -i\hbar\frac{\partial\Psi(x,t)}{\partial x}$$

These definitions generalize immediately to higher dimensions:

$$\hat{x}_i = x_i\,, \qquad \hat{p}_i = -i\hbar\partial_i$$

so for instance $\hat{p}_z = -i\hbar\partial_z$ with $\hat{z} = z$, and so on. More generally, if classically my observable is a function of x and p, the corresponding quantum operator is just given by replacing all the x's and p's with their operators $\hat{x}, \hat{p}$. Some other operators you may have learned about:

- The Hamiltonian, or total energy operator $\hat{H}$, which for a massive particle moving in a potential $U(x)$ is equal to:

  $$\hat{H} = \frac{\hat{p}^2}{2m} + U(\hat{x}) = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(\hat{x})$$

  In 3 dimensions, the ∂_x^2 is replaced by the Laplacian, ∇^2.

- The angular momentum operators $\hat{L}_i$, which in Cartesian coordinates are

  $$\begin{aligned}\hat{L}_x &= \hat{y}\hat{p}_z - \hat{z}\hat{p}_y = -i\hbar\,(\hat{y}\partial_z - \hat{z}\partial_y)\\ \hat{L}_y &= \hat{z}\hat{p}_x - \hat{x}\hat{p}_z = -i\hbar\,(\hat{z}\partial_x - \hat{x}\partial_z)\\ \hat{L}_z &= \hat{x}\hat{p}_y - \hat{y}\hat{p}_x = -i\hbar\,(\hat{x}\partial_y - \hat{y}\partial_x)\end{aligned}$$

One way we use the system's wavefunctions to gain information about observables is by computing averages, or expectation values, of the observables. The expectation value of an observable O at time t for a system in a state described by a wavefunction Ψ is given by

$$\langle O\rangle = \int_{-\infty}^{\infty}\Psi^*\hat{O}\Psi\,dx$$

For instance, I can ask: what is the expectation value $\langle x\rangle$ for a particle trapped in a box of length L in its ground state? In this case I would use the wavefunction you derived in PHYS 200 for this system, that $\Psi(x,t) = \sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L})e^{-iE_1t/\hbar}$ (where E_1 is the ground state energy of the particle) for $0 \le x \le L$ and $\Psi(x,t) = 0$ outside of the box, to compute

$$\begin{aligned}\langle x\rangle &= \int_{-\infty}^{\infty}\Psi^*\hat{x}\Psi\,dx = \int_0^L\left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)e^{+iE_1t/\hbar}\right)\hat{x}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{\pi x}{L}\right)e^{-iE_1t/\hbar}\right)\,dx\\ &= \frac{2}{L}\int_0^L x\sin^2\left(\frac{\pi x}{L}\right)\,dx = \frac{L}{2}\end{aligned}$$

This means that if I make a bunch of measurements of the particle's location and average them, on average I'll find the particle in the center of the box.

Another question I can ask is: what is the uncertainty, or standard deviation, of an observable O? In other words, if I make many measurements of the observable O associated to the operator $\hat{O}$, what is the standard deviation of the distribution of measured values? The uncertainty is,

$$\Delta O = \sqrt{\langle\hat{O}^2\rangle - \langle\hat{O}\rangle^2} \tag{5.1}$$

In other words, I compute the expectation value of $\hat{O}^2$, and the expectation value of $\hat{O}$, subtract the square of the latter from the former and take the square root. For instance, in the example of a particle in a box in its ground state, one would compute that

$$\Delta x = \sqrt{\langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2} = 0.18L$$

So on average, I would find the particle in the center of the box, but the standard deviation of my set of measurements is still pretty large at almost 1/5 the length of the box.

——— *End Lecture 20.*

An important question is: when does a quantum state Ψ have a definite value of an observable? Sometimes it happens that acting with an operator on the wavefunction gives back a constant times the wavefunction again:

$$\hat{O}\Psi = \lambda_0 \Psi\,, \qquad \lambda_0 \text{ a constant} \tag{5.2}$$

When this is the case, we say that the observable is "sharp": there is 100% chance that when we measure $\hat{O}$, we will yield the value λ_0. In particular, there is zero uncertainty in the outcome of the measurement (a nice exercise that I will not assign: verify that this follows from the definition of the uncertainty (5.1)!). On the other hand, if this is *not* true, so that $\hat{O}\Psi \neq (\text{constant})\Psi$, the observable is "fuzzy": there is necessarily some uncertainty in the outcome of the measurement, because there is more than one possibility for what measurement of $\hat{O}$ will return. In this latter case, the best we can do is calculate probabilities for each measurement outcome.

Mathematically, (5.2) is called the eigenvalue equation for the operator $\hat{O}$. This is because as a linear operator, $\hat{O}$ can be represented in some basis by a matrix, which acts on the wavefunction Ψ, a vector in the *Hilbert space*. Then, this equation is identical to the eigenvalue equation for a matrix: it says that when I act with the matrix $\hat{O}$ on a vector Ψ, I get out a constant λ_0 times the vector again, so that Ψ is an eigenstate/eigenvector/eigenfunction of $\hat{O}$ with eigenvalue λ_0. (Really, quantum mechancis is just a whole lot of linear algebra..)

Time evolution The time evolution of the wavefunction $\Psi(x,t)$ of a particle in the presence of a potential $U(x)$ is given by the Schrödinger equation,

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Psi}{\partial x^2} + U(x)\Psi = i\hbar \frac{\partial \Psi}{\partial t}$$

We recognize the left-hand-side of this equation as the Hamiltonian operator $\hat{H}$ acting on the wavefunction Ψ. In fact, the general version of the time-evolution equation for a quantum system described by wavefunction Ψ with Hamiltonian H is,

$$\text{Schrödinger equation:} \qquad \hat{H}\Psi = i\hbar \partial_t \Psi \tag{5.3}$$

You should think of the time-dependent Schrödinger equation (5.3) as the quantum equation of motion satisfied by the state of the system.

We are often interested in the solutions to the Schrödinger equation that have constant, definite energy E. In such cases, the time-dependent wavefunction separates into a time-dependent phase and time-independent spatial wavefunction $\psi(x)$,

$$\Psi(x,t) = \psi(x) e^{-iEt/\hbar} \tag{5.4}$$

Substituting (5.4) into (5.3), the spatial wavefunction $\psi(x)$ can be seen to satisfy the time-*independent* Schrödinger equation (TISE),

$$\text{TISE:} \qquad -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + U(x)\psi(x) = E\psi(x)$$

Such definite-energy states are called *stationary states*, because all probabilities computed in stationary states are time-independent ("stationary" in time), since the phase factor cancels when computing $\Psi(x,t)^*\hat{O}\Psi(x,t) = \psi(x)^*\hat{O}\psi(x)$ as long as $\hat{O}$ itself does not depend on time. Written more succinctly, the TISE equation takes the form

$$\text{TISE:} \qquad \hat{H}\psi(x) = E\psi(x)\,. \tag{5.5}$$

In other words, it is the requirement that acting with the total energy operator $\hat{H}$ on the wavefunction gives back a constant – the energy of the state – times the wavefunction, so that the wavefunction satisfies the eigenvalue equation for the Hamiltonian. A system in a stationary state with energy E has a wavefunction that is an eigenstate of the Hamiltonian operator $\hat{H}$, with eigenvalue equal to the definite energy E of the system. All wavefunctions with definite, "sharp" energy must satisfy the TISE (5.5).

5.2 The quantum version of Noether's theorem

We learned that classically, Noether's theorem maps a symmetry of the Hamiltonian satisfying $\{Q, H\} = 0$ to a constant of motion $dQ/dt = 0$, so that Q is a conserved quantity. What is the analogous statement for a quantum system?

Canonical quantization The answer can be most simply stated using Dirac's prescription of mapping a classical Hamiltonian system to a quantum system, known as *canonical quantization*. The general prescription is as follows: one replaces all classical variables x, p, H, *etc.* with their quantum operators $\hat{x}, \hat{p}, \hat{H}$, *etc*; and replaces the Poisson brackets with the *commutator*:

$$\{f(p,q),\, g(p,q)\}_{\text{classical}} \quad \leftrightarrow \quad -\frac{i}{\hbar}[\hat{f}(\hat{p},\hat{q}),\, \hat{g}(\hat{p},\hat{q})]_{\text{quantum}} \tag{5.6}$$

Let's examine why this rule makes sense with what you've learned about quantum mechanics. Firstly, consider the Poisson bracket between the canonically conjugate coordinates and momenta,

$$\{x_i, p_j\} = \delta_{ij}\,, \qquad \{x_i, x_j\} = \{p_i, p_j\} = 0$$

Following Dirac's prescription, the first set of relations would be replaced by,

$$\{\hat{x}_i, \hat{p}_j\} = \delta_{ij} \quad \longrightarrow \quad -\frac{i}{\hbar}[\hat{x}_i, \hat{p}_j] = \delta_{ij} \quad \Rightarrow \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij} \tag{5.7}$$

Does this make sense? Let's check explicitly by acting with the commutator on some general wavefunction Ψ to compute what $[\hat{x}_i, \hat{p}_j]$ should be:

$$\begin{aligned}[\hat{x}_i, \hat{p}_j]\Psi = (\hat{x}_i\hat{p}_j - \hat{p}_j\hat{x}_i)\Psi &= x_i\left(-i\hbar\partial_j\right)\Psi - \left(-i\hbar\partial_j\right)x_i\Psi \\ &= -i\hbar x_i\partial_j\Psi + i\hbar\left(\delta_{ij}\Psi + x_i\partial_j\Psi\right) \\ &= i\hbar\delta_{ij}\Psi \quad \checkmark\end{aligned}$$

where first we expanded out the definition of the commutator, and then in the second term we used the chain rule to act with ∂_j first on x_i and then on Ψ, and then canceled terms. Similarly, we will have that $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$; these follow from the fact that $x_i x_j = x_j x_i$, and $\partial_x \partial_y = \partial_y \partial_x$. Therefore by explicit computation, following the rule (5.6) indeed yields the following quantum mechanical commutation relations:

$$[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}\,, \qquad [\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0 \tag{5.8}$$

Furthermore, the commutation relations (5.8) are precisely the origin of the famous Heisenberg uncertainty relation! One can show that for two observables A and B, the product of their uncertainties is bounded by their commutator, as[19]

$$\Delta A\, \Delta B \geq \frac{1}{2}\left|\langle [\hat{A}, \hat{B}]\rangle\right|$$

Applying this identity to Δx and Δp, we obtain

$$\Delta x\, \Delta p \geq \frac{1}{2}\,|\langle [\hat{x}, \hat{p}]\rangle| = \frac{1}{2}|\langle i\hbar\rangle| = \frac{\hbar}{2}$$

This exactly reproduces the uncertainty principle $\Delta x \Delta p \geq \hbar/2$, which states that we cannot precisely simultaneously know both the position and momentum of a particle.

Symmetries and Noether's theorem Next, let's examine what the rule (5.6) implies for the time evolution of operators. Recall that the classical equation of motion describing the time evolution of a function $f(x,p)$ is given by the following Poisson bracket with the Hamiltonian (recall (4.31)),

$$\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}\,.$$

Therefore, using the rule (5.6) we expect that the time evolution of an operator $\hat{O}(t)$ in the quantum mechanical system is given by the following commutator with the Hamiltonian,

$$\frac{dO}{dt} = \{O, H\} + \frac{\partial O}{\partial t} \quad \longrightarrow \quad \frac{d\hat{O}}{dt} = -\frac{i}{\hbar}[\hat{O}, \hat{H}] + \frac{\partial \hat{O}}{\partial t}$$

or rearranging a bit,

$$i\hbar\frac{d\hat{O}}{dt} = [\hat{O}, \hat{H}] + i\hbar\frac{\partial \hat{O}}{\partial t} \tag{5.9}$$

Therefore, any operator $\hat{O}$ for which the right-hand-side of (5.9) is zero corresponds to a constant of motion of the quantum system:

$$[\hat{O}, \hat{H}] = -i\hbar\partial_t\hat{O} \quad \Leftrightarrow \quad \hat{O} \text{ a constant of motion} \tag{5.10}$$

In particular, if $\hat{O}$ is independent of time, this is the condition that *a constant of motion $\hat{O}$ commutes with the Hamiltonian.*

So far so good, now how do we see that $\hat{O}$ satisfying (5.10) is related to a symmetry of the quantum system? Suppose our system is described by a wavefunction $\Psi(x,t)$ that satisfies the Schrödinger equation (5.3), $i\hbar\partial_t\Psi = \hat{H}\Psi$. We would consider some operator $\hat{S}$ to be a *symmetry*

[19] This is a general identity that we will not prove here.

of the quantum dynamics if when we transformed the wavefunction by $\hat{S}$, then the transformed wavefunction Ψ' is *also* a solution to the Schrödinger equation:

$$\Psi \to \hat{S}\Psi = \Psi' \quad \text{such that} \quad i\hbar\partial_t\Psi' = \hat{H}\Psi' \quad \Leftrightarrow \quad \hat{S} \text{ a symmetry operator}$$

Let us call $\hat{S}$ a **symmetry operator**. Explicitly, the condition that $\hat{S}$ is a symmetry operator can be written,

$$\begin{aligned} i\hbar\partial_t\left(\hat{S}\Psi\right) &\stackrel{!}{=} \hat{H}\left(\hat{S}\Psi\right) \\ i\hbar\Big((\partial_t\hat{S})\Psi + \hat{S}\partial_t\Psi\Big) = i\hbar(\partial_t\hat{S})\Psi + \hat{S}(\hat{H}\Psi) &= \hat{H}(\hat{S}\Psi) \\ \Rightarrow [\hat{S},\hat{H}]\Psi = -i\hbar\partial_t\hat{S}\,\Psi \quad &\checkmark \end{aligned} \tag{5.11}$$

where in the second line we used the fact that Ψ satisfies the Schrödinger equation (5.3), and then to go to the last line we rearranged terms between the left-hand-side and the right-hand-side and used the definition of the commutator. Therefore, *$\hat{S}$ transforms solutions of the Schrödinger equation onto other solutions if and only if $\hat{S}$ corresponds to a constant of motion.*

Note that there is an important ambiguity in going back and forth between the constant of motion and the symmetry operator. For simplicity, let's restrict the rest of the discussion to the (usual) case where the symmetry operator / constant of motion are independent of time, so that the condition for $\hat{O}$ to be a constant of motion is that $[\hat{O},\hat{H}] = 0$. Suppose we have identified the symmetry operator $\hat{S}$ for some symmetry of the system. As we just proved in (5.11), that symmetry operator commutes with the Hamiltonian, and therefore is a constant of motion. However, let's go the other way around instead: supposed we have first found a constant of motion $\hat{O}$ that commutes with the Hamiltonian. Then, any operator which is a function of only $\hat{O}$ and constants *also* commutes with the Hamiltonian: in particular,

$$\text{If } [\hat{O},\hat{H}] = 0\,, \quad \text{then } \left[\sum_{n=0}^{\infty}\alpha_n\hat{O}^n,\hat{H}\right] = 0$$

While it is always true that $\hat{O}\Psi$ will also satisfy the Schrödinger equation (this is what we proved in (5.11)!), it might be the case that when we identify the symmetry operator $\hat{S}$ that transforms the wavefunction in the way that we expect to correspond to that symmetry, it is such a function of $\hat{O}$ and not just $\hat{O}$ itself. We will see this subtlety play out in our examples below.

This is Noether's theorem for quantum mechanics: every symmetry of the Hamiltonian leads to an associated conserved quantity. To summarize: given a symmetry operator $\hat{S}$ that transforms the wavefunction as $\Psi \to \hat{S}\Psi$ while leaving the Schrödinger equation invariant, we can identify a constant of motion $\hat{S}$ that satisfies $[\hat{S},\hat{H}] = -i\hbar\partial_t\hat{S}$. On the other hand, given a constant of motion $\hat{O}$ that satisfies $[\hat{O},\hat{H}] = -i\hbar\partial_t\hat{O}$, we can identify a corresponding symmetry operator which is some function of $\hat{O}$.

Example: Translations

Let's explore this theorem through examples. Consider a particle in three dimensions subject to a constant potential U, which evidently has translational symmetry. The Hamiltonian is

$$\hat{H} = \frac{\hat{p}_x^2}{2m} + \frac{\hat{p}_y^2}{2m} + \frac{\hat{p}_z^2}{2m} + U\,.$$

From our classical intuition, we expect that this translational symmetry should imply momentum conservation. Indeed, since $[\hat{p}_i, \hat{p}_j] = 0$, we evidently have that momentum is conserved,

$$[\hat{p}_i, \hat{H}] = 0 \quad \Rightarrow \quad \frac{d\hat{p}_i}{dt} = 0$$

How do we show in the quantum language that momentum is the conserved quantity corresponding to this symmetry? We need to identify the symmetry operators $\hat{S}_i$ that enacts the translations. Call these operator $\hat{S}_i(a_i)$: these are by definition the operators that act on the wavefunction and shift all the positions by a constant amount $\vec{a}$:

$$\hat{S}_i(a_i)\Psi(x,t) = \Psi(x_i + a_i, t)$$

Since the Hamiltonian is separately invariant under translations in x, y, and z, we've given one symmetry operator for each translation: $S_x(a_x)$ translates $x \to x + a_x$, $S_y(a_y)$ translates $y \to y + a_y$, and $S_z(a_z)$ translates $z \to z + a_z$.

Clearly, the operators $\hat{p}_i$ are not quite good enough for this job: it is NOT true for instance that $a_x\hat{p}_x\Psi(x) = \Psi(x + a)$. However, as we emphasized in the subtlety above, we expect that the symmetry operator that does the job should be some *function* of the momenta $\hat{p}_i$. It turns out that the correct symmetry operators are,

$$\hat{S}_i(a_i) = e^{ia_i\hat{p}_i/\hbar} \tag{5.12}$$

We can verify this by expanding for small a_i: for instance, considering translations in the x-direction and expanding the exponential in its Taylor series,

$$\hat{S}_x(a_x) = e^{ia_x\hat{p}_x/\hbar} = 1 + \frac{i}{\hbar}a_x\hat{p}_x + \cdots = 1 + \frac{i}{\hbar}(-i\hbar)a_x\partial_x + \ldots$$

Acting on the wavefunction:

$$\begin{aligned}\hat{S}_x(a_x)\Psi(\vec{x},t) &= [1 + a_x\partial_x + \ldots]\,\Psi(\vec{x},t) \\ &= \Psi(\vec{x},t) + a_x\partial_x\Psi(\vec{x},t) + \ldots \\ &= \Psi(x + a_x, y, z, t)\end{aligned}$$

We recognize action of $S_x(a_x)$ as precisely equal to the Taylor series of the wavefunction $\Psi(x+a_x)$ for infinitesimal a_x.

Putting all of this together: the symmetry operator implementing translations by an amount a_i in the $\hat{x}_i$-direction in quantum mechanics is the operator,

$$\hat{S}_i(a_i) = e^{ia_i\hat{p}_i/\hbar}: \quad \hat{S}_i(a_i)\Psi(x,t) = \Psi(x_i + a_i, t)$$

And while it is true that $\hat{S}_i$ itself is a conserved quantity, we would more commonly say that the corresponding conserved quantity is the momentum in that direction,

$$\Leftrightarrow \ \hat{p}_i \text{ is conserved.}$$

——— End Lecture 21.

Generators of continuous symmetries You might have noticed that the symmetry operator (5.12) looks suspiciously like the group element (4.26), where recall that in (4.26) we identified the group element $M(\epsilon)$ implementing a continuous symmetry transformation parameterized by ϵ with the exponential of the *generator* of the group, T, as $M(\epsilon) = e^{\epsilon T}$. Comparing with (5.12), we see that if we identify the constants a_i with the ϵ parameter, and the combination $i\hat{p}_i/\hbar = \partial_i$ with the generator T, it is the same equation. In other words: *in the quantum system, the momentum operator $\hat{p}_i$ generates translations!*

The general statement is as follows: for a continuous symmetry of the quantum system, the symmetry operator $\hat{S}(\epsilon)$ that implements a symmetry transformation parameterized by group parameter ϵ acts on the wavefunction as,

$$\hat{S}(\epsilon)\Psi = \Psi' = \Psi + \epsilon\delta\Psi \tag{5.13}$$

such that if Ψ satisfies the Schrödinger equation, Ψ' also satisfies the Schrödinger equation. Note that this occurs if and only if $[\hat{S}(\epsilon), H] = 0$ (assuming the case that $\hat{S}(\epsilon)$ has no explicit time dependence). The symmetry operator can be written in terms of the conserved quantity / constant of motion $\hat{O}$ as,[20]

$$\hat{S}(\epsilon) = e^{i\epsilon\hat{O}/\hbar}\,. \tag{5.14}$$

The generator of the symmetry group $\hat{T}$ is identified with the the conserved operator $\hat{O}$ as,

$$\text{symmetry generator:} \quad \hat{T} = \frac{i}{\hbar}\hat{O}\,. \tag{5.15}$$

Applying this logic to other continuous symmetries, you can imagine how it will go: we will find that the Hamiltonian operator $\hat{H}$ generates time translations, with systems that are invariant under time translations having conserved energy; the angular momentum operators $\hat{L}_n$ generate rotations about the n-axis, with systems that are invariant under rotations having conserved angular momentum; and so on. We can summarize some of the main results:

Symmetry	Action	Symmetry Operator	Generator	Conserved Quantity
translations	$\Psi(\vec{x}) \to \Psi(\vec{x}+\vec{a})$	$e^{\frac{i}{\hbar}a_i\hat{p}_i}$	$\frac{i}{\hbar}\hat{p}_i$	$\hat{p}_i = -i\hbar\partial_i$
time translations	$\Psi(t) \to \Psi(t+a)$	$e^{-\frac{i}{\hbar}a\hat{H}}$	$\frac{i}{\hbar}\hat{H}$	$\hat{H} = i\hbar\partial_t$
rotations	$\Psi(\vec{x}) \to \Psi(R_{\hat{n}}(\theta)\cdot\vec{x})$	$e^{\frac{i}{\hbar}\theta\hat{L}_n}$	$\frac{i}{\hbar}\hat{L}_n$	$\hat{L}_n = \left(\hat{\vec{x}}\times\hat{\vec{p}}\right)_n$

Table 1: Summary of some common symmetry transformations of quantum systems.

In this table we identified the time evolution operator $i\hbar\partial_t$ with the Hamiltonian $\hat{H}$, due to the Schrödinger equation (5.3). You will also recognize the rotation generators $T_n = \frac{i}{\hbar}\hat{L}_n$ as precisely the generators we identified in (4.27). The statement is that the matrices given in (4.27) furnish a 3-dimensional representation of the angular momentum operators. We will revisit this statement in our example of the Hydrogen atom below.

——— *End Lecture 22.*

[20] Note the other important requirement that $\hat{S}^\dagger\hat{S} = \mathbb{1}$: this is so that the symmetry transformation does not affect the computation of physical quantities:

$$\int \Psi^\dagger\Psi \to \int \Psi^\dagger\hat{S}^\dagger\hat{S}\Psi = \int \Psi^\dagger\Psi\,.$$

This is encapsulated by *Wigner's theorem*, which states that the symmetry operator $\hat{S}(\epsilon)$ is either *unitary* or *anti-unitary*. This is evidently true of (5.14) when $\hat{O}$ is Hermitian.

5.3 Symmetries imply degeneracies

Suppose a system with Hamiltonian $\hat{H}$ possesses a symmetry implemented by the symmetry operator $\hat{S}$, so $[\hat{S}, \hat{H}] = 0$. Consider the stationary states of this system, labeled by their definite energies E_n and spatial wavefunctions ψ_n:

$$\hat{H}\psi_n(x) = E_n\psi_n(x)$$

Now, act with the symmetry operator $\hat{S}$ on both sides of this equation, and use the fact that $[\hat{S}, \hat{H}] = 0$ to commute $\hat{S}$ through $\hat{H}$ on the left-hand-side:

$$\begin{aligned} \hat{S}\hat{H}\psi_n(x) &= \hat{S}E_n\psi_n(x) \\ \hat{S}\hat{H}\psi_n(x) - [\hat{S}, \hat{H}]\psi_n(x) &= E_n\left(\hat{S}\psi_n(x)\right) \\ \hat{H}\left(\hat{S}\psi_n(x)\right) &= E_n\left(\hat{S}\psi_n(x)\right) \end{aligned}$$

This equation says that if $\psi_n(x)$ is a stationary state, the transformed wavefunction $\hat{S}\psi_n(x)$ is *also* a stationary state with the *same* energy E_n. Therefore, the energy spectrum of the theory has a **degeneracy**: both the wavefunctions ψ_n and $\hat{S}\psi_n$ have the same energy.

In linear algebra terms, the stationary state wavefunctions $\psi_n(x)$ are the eigenstates of the Hamiltonian operator $\hat{H}$ (as they satisfy the eigenvalue equation with eigenvalue equal to the energy E_n of the state). The relation $[\hat{S}, \hat{H}] = 0$ implies that one can find a basis of eigenstates of the symmetry operator $\hat{S}$ that are simultaneously stationary states (eigenstates of $\hat{H}$). In particular, when there is a symmetry we can label the stationary states by both the subscript n labeling their energy E_n, and a subscript (say) m labeling their $\hat{S}$ eigenvalue, say s_m:

$$\hat{H}\psi_{nm}(x) = E_n\psi_{nm}(x)\,, \qquad \hat{S}\psi_{nm}(x) = s_m\psi_{nm}(x)$$

We say that the operators $\hat{H}$ and $\hat{S}$ can be *simultaneously diagonalized*. This also means that if I start at $t = 0$ in a stationary state with definite $\{E_n, s_m\}$, for all time the state will continue to have the same definite values of $\{E_n, s_m\}$, since stationary states time evolve just with the complex exponential factor $e^{-iE_nt/\hbar}$.

Equations like this hopefully look familiar from your quantum class, where you learned that the hydrogen atom has a degenerate energy spectrum, with wavefunctions labeled by the principal quantum number n specifying the energy E_n of the state, as well as 3 other quantum numbers ℓ, m, s labeling the orbital, magnetic, and spin quantum numbers. Because the labels ℓ, m, s can run over a variety of values for a given n, it works out that each energy level n with energy E_n has a degeneracy of $2n^2$ different possible states $\psi_{n\ell ms}$ with the same energy. Here, we're learning that the *reason* for this degeneracy is the symmetry of the hydrogen atom Hamiltonian, where each of the ℓ, m, s quantum numbers originates from a symmetry of the hydrogen atom Hamiltonian, and the allowed values of these quantum numbers correspond to the different eigenstates of the symmetry operators. We'll explore this in the following example.

Example: Symmetry and degeneracy in the hydrogen atom

Consider the hydrogen atom, which consists of an electron and proton interacting via the Coulomb potential, $U(r) = -ke^2/r$. The Hamiltonian for this system is,

$$\hat{H} = \frac{1}{2m_e}\left(\hat{p}_x^2 + \hat{p}_y^2 + \hat{p}_z^2\right) + U(\hat{r}) = -\frac{\hbar^2}{2m_e}\nabla^2 - \frac{ke^2}{\hat{r}}\,.$$

What are the symmetries of this system? This is a 2-body system with a central potential that depends only on the radial distance between the electron and proton, and therefore possesses (1) rotational symmetry and (2) time-translational symmetry. We thus expect that angular momentum and energy are conserved. Indeed, we will presently verify that $[\hat{L}_x, \hat{H}] = [\hat{L}_y, \hat{H}] = [\hat{L}_z, \hat{H}] = 0$. We will then use this observation to show that the hydrogen atom stationary state wavefunctions should be labeled by quantum numbers n, ℓ and m. (We will unfortunately not have space to explain the origin of the spin quantum number $s = \frac{1}{2}$; this one is slightly more subtle and a great opportunity for a project!)

First, let's convince ourselves that $[\hat{L}_i, \hat{H}] = 0$ for each of $i = x, y, z$, so that all three components of the angular momentum are indeed symmetries of this quantum system. Consider the commutator $[\hat{L}_z, \hat{H}]$. To evaluate it, it will be helpful to to recall the canonical commutation relations $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ with $[\hat{x}_i, \hat{x}_j] = [\hat{p}_i, \hat{p}_j] = 0$ (recall (5.7)), and to use some identities involving commutators; in particular,

$$[A + B, C] = [A, C] + [B, C] \tag{5.16}$$
$$[AB, C] = A[B, C] + [B, C]A \tag{5.17}$$
$$[f(\vec{x}), p_i] = i\hbar \frac{\partial f}{\partial x_i} \tag{5.18}$$
$$[\hat{x}_i, g(\vec{p})] = i\hbar \frac{\partial g}{\partial p_i} \tag{5.19}$$

as well as the fact that the commutator is antisymmetric, $[A, B] = -[B, A]$. We compute

$$\begin{aligned}[\hat{L}_z, \hat{H}] &= [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x, \hat{H}] = [\hat{x}\hat{p}_y, \hat{H}] - [\hat{y}\hat{p}_x, \hat{H}] \\ &= \hat{x}[\hat{p}_y, \hat{H}] + [\hat{x}, \hat{H}]\hat{p}_y - \hat{y}[\hat{p}_x, \hat{H}] - [\hat{y}, \hat{H}]\hat{p}_y\end{aligned}$$

where in the first line we used linearity of the commutator (5.16), and in the second line we used the identity (5.17) twice. Now, each term in the above expression takes the form of either (minus) (5.18), or (5.19), since the kinetic terms in the Hamiltonian depend only on the momenta, and the potential is only a function of $r = \sqrt{x^2 + y^2 + z^2}$. The only nonzero commutators come from terms where a position and its conjugate momentum are involved. We need to compute, in particular,

$$\begin{aligned}[\hat{p}_i, 1/\hat{r}] &= -i\hbar \frac{\partial}{\partial x_i}(x^2 + y^2 + z^2)^{-1/2} = \frac{i\hbar}{2r^3} 2x_i = i\hbar \frac{x_i}{r^3} \\ [\hat{x}_i, \hat{p}_j^2] &= i\hbar \frac{\partial}{\partial p_i} \hat{p}_j^2 = 2i\hbar p_i \delta_{ij}\end{aligned}$$

where in the first line we used (5.18), and in the second we used (5.19). These allow us to finish the computation,

$$\begin{aligned}[\hat{L}_z, \hat{H}] &= \hat{x}\left[\hat{p}_y, -\frac{ke^2}{\hat{r}}\right] + \left[\hat{x}, \frac{\hat{p}_x^2}{2m_e}\right]\hat{p}_y - \hat{y}\left[\hat{p}_x, -\frac{ke^2}{\hat{r}}\right] - \left[\hat{y}, \frac{\hat{p}_y^2}{2m_e}\right]\hat{p}_x \\ &= -i\hbar ke^2 \hat{x}\hat{y}\frac{1}{r^3} + \frac{1}{2m_e} i\hbar 2\hat{p}_x\hat{p}_y + i\hbar ke^2 \hat{y}\hat{x}\frac{1}{r^3} - \frac{1}{2m_e} i\hbar 2\hat{p}_y\hat{p}_x = 0\end{aligned}$$

As expected, all the terms canceled out so that $\hat{L}_z$ commutes with the Hamiltonian, implying that the z-component of angular momentum is conserved. The commutators for $\hat{L}_x$ and $\hat{L}_y$ can be obtained with almost identical computations, with the final result that

$$[\hat{L}_i, \hat{H}] = 0\,, \qquad i = x, y, z\,. \tag{5.20}$$

Now, let's apply what we learned about degeneracy. Consider a stationary state wavefunction with energy E_n. Due to (5.20), can should be able to label these wavefunctions by their $\hat{L}_z$, $\hat{L}_x$, and $\hat{L}_y$ eigenvalues, right?

The answer is almost, but there's a wrinkle: even though individually each of $\hat{L}_x, \hat{L}_y$, and $\hat{L}_z$ commute with the Hamiltonian (so that they are all definitely symmetries!) they do *not* commute with each other. It is a linear algebra fact that we can only simultaneously diagonalize operators that commute with one another – meaning we can only simultaneously have a state with definite values for a set of observables whose operators mutually commute. We can check that our operators $\hat{L}_i$ are *not* mutually commuting by, for instance, computing the commutator $[\hat{L}_x, \hat{L}_y]$. The computation proceeds as follows,[21]

$$[\hat{L}_x, \hat{L}_y] = [\hat{y}\hat{p}_z - \hat{z}\hat{p}_y, \hat{z}\hat{p}_x - \hat{x}\hat{p}_z] = [\hat{y}\hat{p}_z, \hat{z}\hat{p}_x] - [\hat{y}\hat{p}_z, \hat{x}\hat{p}_z] - [\hat{z}\hat{p}_y, \hat{z}\hat{p}_x] + [\hat{z}\hat{p}_y, \hat{x}\hat{p}_z]$$

When we expand out each of these terms using (5.17), we will only get something nonzero when a $\hat{y}$ is being commuted with a $\hat{p}_y$, or a $\hat{z}$ with a $\hat{p}_z$, or a $\hat{x}$ with a $\hat{p}_x$. In the first term, this means picking out only the piece that looks like $\hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x$; in the second and third terms there is no nonzero contribution; while in the last term it means picking out the piece that looks like $\hat{x}[\hat{z}, \hat{p}_z]\hat{p}_y$. So, we can evaluate

$$[\hat{L}_x, \hat{L}_y] = \hat{y}[\hat{p}_z, \hat{z}]\hat{p}_x + \hat{x}[\hat{z}, \hat{p}_z]\hat{p}_y = -i\hbar\hat{y}\hat{p}_x + i\hbar\hat{x}\hat{p}_y = i\hbar\hat{L}_z$$

Continuing with similar computations, one can check that these three operators satisfy,

$$[\hat{L}_x, \hat{L}_y] = i\hbar\hat{L}_z\,, \qquad [\hat{L}_y, \hat{L}_z] = i\hbar\hat{L}_x\,, \qquad [\hat{L}_z, \hat{L}_x] = i\hbar\hat{L}_y\,. \tag{5.21}$$

It is no accident that the relations (5.21) are precisely the same commutation relations (up to some $i\hbar$'s) satisfied by the $SO(3)$ rotation group generators $\hat{T}_i$ that we found in (4.28) – after all, according to Table 1 we're supposed to identify the rotation group generators with the angular momentum operators as $\hat{T}_i = \frac{i}{\hbar}\hat{L}_i$!

Back to the main point: definite energy states of the Hamiltonian can be labeled by a set of *maximally commuting* operators, that commute both amongst themselves and with the Hamiltonian. For the angular momentum group $SO(3)$ with the three generators $\hat{L}_i$ satisfying (5.21), it turns out that we can choose at most two linearly independent combinations of these operators that each both commute with one another, and also commute with $\hat{H}$. This maximally commuting set is conventionally taken to be the operator $\hat{L}_z$ along with the *total angular momentum-squared operator* $\hat{L}^2$:

$$\hat{L}_z\,, \quad \hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 \qquad \text{satisfy} \qquad [\hat{L}_z, \hat{L}^2] = [\hat{L}_z, \hat{H}] = [\hat{L}^2, \hat{H}] = 0\,.$$

Exercise 5.1

Show (1) that $\hat{L}^2$ commutes with the hydrogen atom Hamiltonian $\hat{H}$, and (2) that $\hat{L}^2$ and $\hat{L}_z$ commute with each other. You are allowed to use the relations $[\hat{L}_i, \hat{H}] = 0$ for $i = x, y, z$ and (5.21) proven in the notes.

We conclude that the Hydrogen atom wavefunctions with definite (sharp) energy E_n *also* have definite (sharp) values of their total angular momentum L^2, and z-component of angular

[21] This is completely identical to the exercise we did in (4.32), because the Poisson bracket and commutator satisfy exactly the same properties and same canonical relations, except that now there are extra $i\hbar$'s hanging around due to the canonical quantization rule (5.6).

momentum L_z. These lead to two additional quantum numbers conventionally called ℓ and m, which label the allowed values of each of these observables,

$$\hat{H}\psi_{n\ell m} = E_n\psi_{n\ell m}\,, \qquad \hat{L}^2\psi_{n\ell m} = f_L(\ell)\psi_{n\ell m}\,, \qquad \hat{L}_z\psi_{n\ell m} = f_z(m)\psi_{n\ell m}\,. \tag{5.22}$$

Here E_n is the energy of the state, $f_L(\ell)$ is its total angular momentum squared, and $f_z(m)$ is its z-component of angular momentum.

Each of $\hat{H}$, $\hat{L}^2$, and $\hat{L}_z$ are differential operators, and so (5.22) represents 3 differential equations that allow one to solve for $\psi_{n\ell m}(x,y,z)$ – although as you might expect due to the spherical symmetry of the problem, it is actually easiest to solve these equations in spherical coordinates for $\psi_{n\ell m}(r,\theta,\phi)$. Solving these equations also fixes the form of $f_L(\ell)$ and $f_z(m)$. The result can be stated as,

$$\hat{H}\psi_{n\ell m} = E_n\psi_{n\ell m}\,, \qquad \hat{L}^2\psi_{n\ell m} = \hbar^2\ell(\ell+1)\psi_{n\ell m}\,, \qquad \hat{L}_z\psi_{n\ell m} = \hbar m\,\psi_{n\ell m} \tag{5.23}$$

where n can be any positive integer, $n = 1, 2, \ldots, \infty$; given a value of n, ℓ runs from 0 to $n-1$ as $\ell = 0, 1, 2, \ldots, n-1$; and given a value of ℓ, m runs from $-\ell$ to ℓ as $m = -\ell, -\ell+1, \ldots, \ell-1, \ell$, leading to the n^2 degeneracy of the n'th level of the hydrogen atom (before the electron's spin is taken into account).

Discrete symmetries have consequences: revisiting parity

Back in Section 2 we introduced the operation of *parity*. Recall that parity is a $\mathbb{Z}_2$ transformation that acts to reflect $\vec{x} \to -\vec{x}$. Let's call the symmetry operator enacting the transformation $\hat{P}$, which acts on the wavefunction as,

$$\Psi(\vec{x}) \quad \to \quad \hat{P}\Psi(\vec{x}) = \Psi(-\vec{x})\,.$$

This operation is an example of the discrete $\mathbb{Z}_2$ group of order 2, since acting twice with $\hat{P}$ just gives back the original wavefunction ($\hat{P}^2 = 1$, the identity operator). Therefore, states that have definite parity are either even or odd under a parity transformation:

$$\text{If } \Psi \text{ has definite parity:} \qquad \hat{P}\Psi(\vec{x}) = \pm\Psi(\vec{x})\,. \tag{5.24}$$

More formally, since $\hat{P}^2 = 1$, the eigenvalues of the parity operator are ± 1,[22] so states that are eigenvectors of the parity operator are either even (with eigenvalue $+1$) or odd (with eigenvalue -1) under the transformation.

So, how does our discussion in this section apply to a discrete transformation? Going back to Table 1, we still have a well-defined symmetry operator (the operator we are calling $\hat{P}$), and when parity is a symmetry of the system this operator commutes with the Hamiltonian as $[\hat{P}, \hat{H}] = 0$. This is true when the Hamiltonian depends on only even powers of the momenta and coordinates – see the aside below for the explanation for why $[\hat{P}, \hat{p}_i^2] = [\hat{P}, \hat{x}_i^2] = 0$. The difference with the continuous case is that there is no sense in which we can write the symmetry operator as the exponential of some generator which we interpret as a conserved observable of the system. But this is ok! Parity is a perfectly good symmetry of many standard quantum systems – including the free particle, particle in a box, quantum harmonic oscillator, hydrogen atom – and has real consequences.

[22] A simple proof of this statement: Consider an eigenvector $\vec{v}$ of the parity operator with eigenvalue p, so that $\hat{P}\vec{v} = p\vec{v}$. Act with $\hat{P}$ again on both sides and use $\hat{P}^2 = 1$: $\hat{P}^2\vec{v} = \vec{v} = p(\hat{P}\vec{v})$. Comparing the first and second equations, the eigenvalue p must satisfy $p = 1/p$, or $p^2 = 1$, which sets $p = \pm 1$.

Aside: To see that $\hat{P}$ commutes with the Hamiltonian when the Hamiltonian depends on only even powers of coordinates and momenta, we have to first understand how it acts on the coordinates and momenta. This is a little bit more subtle than just using our classical intuition to say that $\hat{P}x$ should equal $-x$; the correct way to define the action of a symmetry operator in quantum mechanics is by how it acts on the states (which is what we did in requiring that $\Psi(\vec{x}) \to \Psi(-\vec{x})$), and then to make sure it acts as expected on expectation values, in this case to take $\langle\hat{x}\rangle \to -\langle\hat{x}\rangle$. So, we need to check that after applying the transformation to the wavefunction, $\langle\hat{x}\rangle$ transforms as,

$$\langle\hat{x}\rangle = \int \Psi^\dagger \hat{x} \Psi \, dx \quad \to \quad \int (\hat{P}\Psi)^\dagger \hat{x} (\hat{P}\Psi) \, dx = \int \Psi^\dagger (\hat{P}^\dagger \hat{x} \hat{P}) \Psi \, dx \,. \tag{5.25}$$

$\hat{P}$ is a unitary operator ($\hat{P}^\dagger = \hat{P}^{-1}$, see the footnote around (5.14)), and actually since $\hat{P}^2 = 1$, we have that $\hat{P}^\dagger = \hat{P}^{-1} = \hat{P}$. The action of $\hat{P}$ on the operator $\hat{x}$ is therefore set by (5.25) to,

$$\langle\hat{P}\hat{x}\rangle = -\langle\hat{x}\rangle \quad \Leftrightarrow \quad \hat{P}\hat{x}\hat{P} = -\hat{x} \tag{5.26}$$

$\hat{P}$ does *not* commute with $\hat{x}$; multiplying both sides of this equation by $\hat{P}$ on the right, using $\hat{P}^2 = 1$, and rearranging yields

$$\hat{P}\hat{x}\hat{P} = -\hat{x} \quad \Rightarrow \quad \hat{P}\hat{x}\hat{P}^2 = -\hat{x}\hat{P} = \hat{P}\hat{x} \quad \Rightarrow \quad [\hat{P}, \hat{x}] = \hat{P}\hat{x} - \hat{x}\hat{P} = -2\hat{x}\hat{P} \neq 0 \,.$$

But, as expected $\hat{P}$ *does* commute with $\hat{x}^2$, since the two minus signs cancel:

$$[\hat{P}, \hat{x}^2] = \hat{x}[\hat{P}, \hat{x}] + [\hat{P}, \hat{x}]\hat{x} = -2\hat{x}^2\hat{P} - 2\hat{x}\hat{P}\hat{x} = -2\hat{x}^2\hat{P} + 2\hat{x}^2\hat{P} = 0 \,.$$

For exactly the same reasons, parity does not commute with the momentum operator, but does commute with the momentum operator squared: $\hat{P}\hat{p}_i\hat{P} = -\hat{p}_i$ with $[\hat{P}, \hat{p}_i^2] = 0$. *End aside.*

In particular, our entire discussion of degeneracy still applies: If a system is parity-symmetric so that there is an operator $\hat{P}$ acting as $\Psi(\vec{x}) \to \Psi(-\vec{x})$ that commutes with the Hamiltonian as $[\hat{P}, \hat{H}] = 0$, then the Hamiltonian and parity operators can be simultaneously diagonalized, which implies that the stationary states of the system can be taken to have definite even or odd parity as in (5.24). You can then think of the parity of the wavefunction to have its own quantum number / charge, which is either ± 1 (corresponding to either even or odd parity). Then, if we start at time $t = 0$ in a state of definite parity, the time-evolved state will continue to have the same definite parity. This principle is called *conservation of parity*. (In this sense, continuous symmetries aren't the only ones that can be conserved in quantum systems!)

For example, consider a particle in a box. The Hamiltonian within the box $\hat{H} = \hat{p}^2/(2m)$ is parity-symmetric, and therefore the stationary state wavefunctions will have definite even or odd parity. You can verify that the wavefunctions satisfy

$$\psi_n \xrightarrow{\hat{P}} (-1)^{n-1}\psi_n$$

so that the ground state is even under reflection about the center of the box, the first excited state is odd, and so on. Similarly, in the hydrogen atom, you can verify that both $[\hat{P}, \hat{H}] = 0$ and $[\hat{P}, \hat{L}_i] = 0$, so the stationary state wavefunctions $\psi_{n\ell m}$ also have definite parity. It turns out that the hydrogen atom wavefunctions satisfy

$$\psi_{n\ell m} \xrightarrow{\hat{P}} (-1)^\ell \psi_{n\ell m}$$

For more discussion on the non-conservation of parity in particle physics, I recommend the discussion in the Feynman Lectures at this link.

Exercise 5.2

An electron in the hydrogen atom is in the following state,

$$\psi_{n\ell m}(r,\theta,\phi) = \frac{1}{8\sqrt{\pi a_0^3}} \frac{r}{a_0} e^{-r/(2a_0)} \sin\theta e^{i\phi} .$$

(a) Determine the values of E_n, ℓ, and m for this state, by applying (5.23) with the operators $\hat{H}$, $\hat{L}^2$ and $\hat{L}_z$ given in spherical coordinates as,

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \left(\frac{2}{r}\right)\frac{\partial}{\partial r} + \frac{1}{r^2}\left[\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \csc^2\theta\frac{\partial^2}{\partial\phi^2}\right]\right) - \frac{ke^2}{r}$$

$$\hat{L}^2 = -\hbar^2\left(\frac{\partial^2}{\partial\theta^2} + \cot\theta\frac{\partial}{\partial\theta} + \csc^2\theta\frac{\partial^2}{\partial\phi^2}\right)$$

$$\hat{L}_z = -i\hbar\frac{\partial}{\partial\phi}$$

Note that the Bohr radius a_0 and Coulomb's constant k are related as $a_0 = \frac{\hbar^2}{mke^2}$.

(b) Upon measurement, what is the observed value of L_z for this atom? What is the parity of this state?

——— *End Lecture 23.*

6 Spontaneous Symmetry Breaking

Possibly helpful resources:

- As usual, Cambridge professor David Tong has some nice lecture notes on symmetry breaking, available here.
- A very nice and comprehensive introduction to the topic appears in Section 2 of the lectures notes available here.
- A more advanced but classic reference is Sydney Coleman's *Aspects of Symmetry*. Most of the discussion therein involves field theory.
- Kunstatter-Das has a purely conceptual discussion in Section 2.4.3.
- For a comprehensive reference on the Ising model in statistical mechanics, I recommend sections 1.1-1.2 of these notes on Statistical Field Theory. We will largely follow this reference in our discussion of this model.

When considering symmetry breaking, you might have different scenarios in mind. One type of symmetry breaking is an *explicit* breaking: you add a term to the Lagrangian or Hamiltonian of your system that breaks a symmetry that the system had before you added that term. For example, the hydrogen atom Hamiltonian is invariant under rotational symmetry. Putting this system in a magnetic field explicitly breaks some of this symmetry; in particular, this adds the term $\hat{H} = -\vec{\mu}\cdot\vec{B}$, where $\vec{\mu} = e\vec{L}/(2m_e)$ is the magnetic dipole momentum of the electron, so that depending on the direction of the magnetic field not all the angular momentum operators will commute with the perturbed Hamiltonian. This breaks part of the degeneracy of the hydrogen atom, causing the energies to shift in what is known as the Zeeman effect. Still, explicit symmetry breaking can be useful if the breaking is small. For instance if the magnetic field applied to the hydrogen atom is small $|B| \ll 1$, the $2\ell+1$ degenerate states only get slightly split, and we can understand the approximate degeneracy as coming from the underlying rotational symmetry of the un-perturbed system.

Another type of symmetry breaking is known as *spontaneous symmetry breaking*. This moniker is a bit of a misnomer, since the symmetry is not really broken but just hidden. Spontaneous symmetry breaking (SSB) is what we call the phenomenon in which a theory is invariant under a symmetry, but the ground state (lowest energy state) is not. In general, there is no reason why a symmetry of the Hamiltonian of a quantum-mechanical system should also result in a symmetry of the ground state. Or analogously, why a symmetry of the equations of motion should also result in a symmetry of the solutions. The underlying equations of motion / Lagrangian / Hamiltonian preserve the symmetry, but the actual state of the system does not, so from the low-energy perspective the symmetry is "hidden". For the rest of this section, we will exclusively discuss this scenario of spontaneous symmetry breaking, which is extremely useful for characterizing the states of systems.

The classic example is that of the Heisenberg ferromagnet: an infinite array of spin-1/2 magnetic dipoles with nearest-neighbor interactions that cause neighboring dipoles to want to align. The Hamiltonian is rotationally invariant, but the ground state is not; it is a state in which all the dipoles are aligned in some particular arbitrary direction. For an infinite ferromagnet, this ground state is infinitely degenerate, and labeled by the direction in which the spins point: any state with all the spins aligned in the same direction is a ground state of the system. If you lived inside the ferromagnet, you would have a hard time detecting the rotational symmetry in the laws of physics, since as far as you can tell that rotational symmetry is broken.

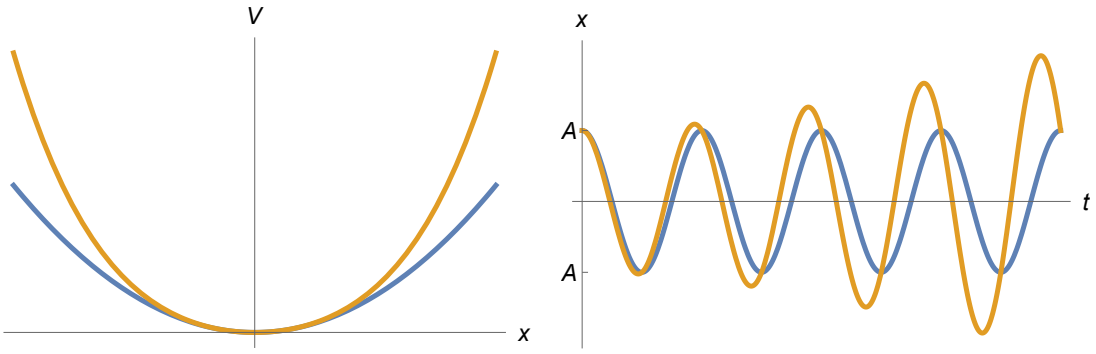


Figure 14: Left: The harmonic oscillator potential $V = \frac{1}{2}m\omega^2 x^2$ (blue), plus a small quartic term $\delta V = \frac{\lambda}{4}x^4$ (in orange). Right: the solution $x(t)$ for $\lambda = 0$ (blue), versus small λ (orange).

6.1 Discrete symmetry breaking in a classical system

Let's consider first an example of spontaneous symmetry breaking in classical physics. Consider a particle of mass m in a harmonic oscillator potential, where we add a small quartic term $V \sim x^4$ to the potential so that the oscillator is anharmonic. The Lagrangian is,

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2 - \frac{\lambda}{4}x^4 \tag{6.1}$$

where λ is a parameter that measures the anharmonicity. In particular, for $\lambda = 0$ this is the standard Lagrangian for a simple harmonic oscillator; there is a minimum of the potential at $x = 0$, and the solutions oscillate around $x = 0$ as (suppose we take as initial conditions $x(0) = A$, $\dot{x}(0) = 0$),

$$x_0(t) = A\cos(\omega t)\,.$$

For nonzero λ, there is still a minimum of the potential at $x = 0$, but to find the form of the solution we need to solve the Euler-Lagrange equations with λ included,

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \quad \Rightarrow \quad -\omega^2 x - \frac{\lambda}{m}x^3 = \ddot{x} \tag{6.2}$$

Perturbatively, for small λ the solutions to these equations take the form[23]

$$\begin{aligned} x(t) &= x_0(t) + \lambda x_1(t) + \dots \\ x_1(t) &= -\frac{A^3}{8m\omega^2}\left(3\omega t\sin(\omega t) + \frac{1}{4}\left(\cos(\omega t) - \cos(3\omega t)\right)\right) \end{aligned}$$

where we neglect terms of order λ^2 and higher, assuming $\lambda \ll 1$. You can verify in particular that the solution $x_0(t)+\lambda x_1(t)$ satisfies the equations of motion (6.2) to order λ. These solutions describe motion which is approximately harmonic, but with an amplitude that grows with time (see Figure 14).

The Lagrangian (6.1) enjoys a $\mathbb{Z}_2$ parity symmetry, since it is invariant under taking $x \to -x$. The anharmonic oscillator solutions we just described respect this symmetry.

[23] This problem is treated perturbatively in Section 6.3.1 of *Classical Dynamics: A Contemporary Approach*, found here. The exact solution can be written in terms of elliptic integrals.

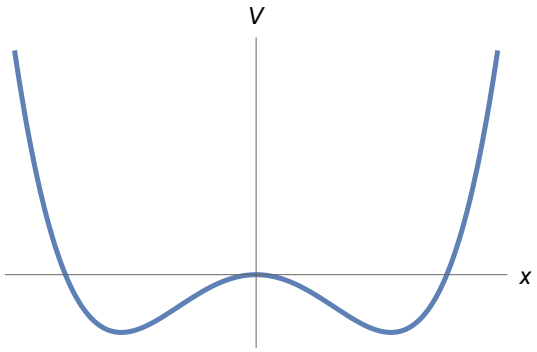


Figure 15: The double well potential $V = -\frac{1}{2}m|\omega^2|x^2 + \frac{\lambda}{4}x^4$.

On the other hand, what if we consider allowing the x^2 term to have the opposite sign, with $\omega^2 < 0$? This system describes a particle moving in the following potential,

$$V(x) = -\frac{1}{2}m|\omega^2|x^2 + \frac{\lambda}{4}x^4\,. \tag{6.3}$$

Now $x = 0$ is a local maximum rather than a local minimum of the potential (see Figure 15), which looks like a double well with two minima at $x = \pm\sqrt{\frac{-m\omega^2}{\lambda}} \equiv v_\pm$. (Yes this seems a little physically strange – literally we are studying a particle moving in a harmonic potential with negative spring constant – but there are many examples in nature where such a potential describes the physics of a system! We will study such an example below.)

The solutions of this system will correspond to approximately harmonic motion executed about either the left minimum or the right minimum. To study the system of a particle moving in this double-well potential potential, it is useful to rewrite the potential as

$$V(x) = \frac{\lambda}{4}\left(x^2 - v_\pm^2\right)^2 + \text{constant}$$

which only shifts the Lagrangian by an unimportant constant. Then, we can expand near the ground state at $x = v_\pm$ by defining the shifted coordinate $\tilde{x}(t)$,

$$x(t) = \tilde{x}(t) + v_\pm$$

in terms of which the potential takes the form,

$$V(\tilde{x}) = \lambda v_\pm \tilde{x}^2 + \lambda v_\pm \tilde{x}^3 + \frac{\lambda}{4}\tilde{x}^4 + \text{constant} \tag{6.4}$$

This describes a particle of mass m (note that $\lambda v_\pm^2 = m|\omega^2|$) executing approximately harmonic motion about either the left minimum centered around v_- or the right minimum centered around v_+. But, neither of these ground states is invariant under the $\mathbb{Z}_2$ symmetry of the underlying Lagrangian! The potential (6.4) for the shifted field is *not* invariant under $\tilde{x} \to -\tilde{x}$ due to the cubic term $\tilde{x}^3$. If I was a little ant living at the bottom of one of these potentials, it would be extremely difficult to detect the hidden $x \to -x$ symmetry directly. Instead, the $\mathbb{Z}_2$ symmetry acts to exchange the two ground states, since taking $v_+ \to -v_+$ or $v_- \to -v_-$ exchanges the two degenerate states.

We can summarize the lesson we have learned from this example as: if a $\mathbb{Z}_2$ symmetry is spontaneously broken, it leads to 2 ground states. More generally, this lesson will extend to other types of symmetry breaking patterns; the consequence of the symmetry, when broken, is to generate multiple degenerate ground states.

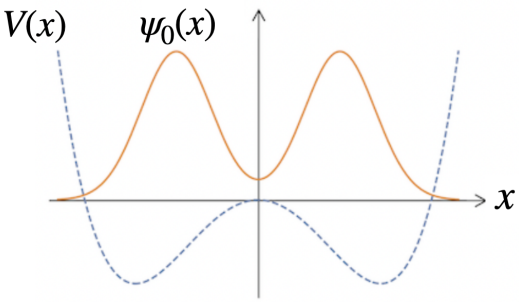


Figure 16: The ground state wavefunction in the double well potential.

6.2 Symmetry breaking in quantum systems

If we consider the quantum mechanical analogue of our previous example, there is a twist: *there is no spontaneous symmetry breaking in quantum mechanical systems with finite degrees of freedom.* The ground state is always invariant under the $\mathbb{Z}_2$ symmetry!

In particular, the Hamiltonian for the double well potential is given by,

$$\hat{H} = \frac{\hat{p}^2}{2m} - \frac{1}{2}m|\omega^2|\hat{x}^2 + \frac{\lambda}{4}\hat{x}^4$$

which as a quadratic function of the momenta and coordinates is invariant under parity, so the parity operator commutes with the Hamiltonian $[\hat{P}, \hat{H}] = 0$ and energy eigenstates are also eigenstates of the $\mathbb{Z}_2$ symmetry. The ground state of this system is depicted in Figure 16, and it is symmetric in $x \to -x$. We can understand this symmetric ground state as the superposition of the wavefunctions that are localized near one or the other minima; roughly,

$$\psi_0(x) \approx \psi_{\text{left}}(x) + \psi_{\text{right}}(x)$$

There is equal probability of finding the particle near the left minimum, or near the right minimum. Another way to say this is that if the particle starts in one minimum, say at $x = v_-$, there is a nonzero probability that it will tunnel through the finite potential barrier in the middle and end up at the other minimum $x = v_+$.

In order to have a quantum system that exhibits spontaneous symmetry breaking, you need *infinite* degrees of freedom. There are multiple ways to see this. From the perspective of double-well problem, the reason the ground state is a superposition of left- and right-localized wavefunctions and that there is a nonzero amplitude to tunnel from the left minimum to the right minimum is because the energy barrier separating the minima is finite. Effectively, you can only have a system with spontaneous symmetry breaking when it costs an infinite amount of energy to tunnel between the two ground state configurations, so that there really are multiple unique ground states.

This requirement of infinite degrees of freedom can be achieved either by upgrading to quantum field theory and considering quantum fields that take values over all of space (in which case the analogous double-well problem *does* have an infinite energy barrier separating the minima), or by considering the thermodynamic limit of quantum systems. The thermodynamic limit for a system of N particles in volume V is defined by taking both the limits $N \to \infty$ and $V \to \infty$ while keeping the ratio N/V fixed, so that intensive quantities like density and temperature remain fixed, but extensive quantities like N and the entropy grow large. (Although practically speaking, many aspects of symmetry-breaking appear in large, but finite, systems, so this is not just a formal exercise!)

Basic notions of SSB Suppose our system exhibits spontaneous symmetry breaking. This means that the Hamiltonian is invariant under the symmetry, commuting with the symmetry operator as,

$$[\hat{S}, \hat{H}] = 0\,,$$

while the ground state ψ_0 of the system is not left invariant by the symmetry transformation. In particular, upon acting with $\hat{S}$ on the ground state, the result is *another* degenerate ground state (state with the same energy) which is inequivalent from the first one:

$$\hat{S}\Psi_0 = \Psi_0'$$

What it means to be able to distinguish between the states Ψ_0 and Ψ_0' is that there is some operator whose expectation values in the two states differs. The statement is: the inequivalence of the ground states is measured by an *order parameter*: a local operator $\hat{O}(x)$ whose expectation value in the ground states is nonzero and different for the inequivalent broken states:[24]

$$\langle \hat{O}(x) \rangle_{\Psi_0} \neq \langle \hat{O}(x) \rangle_{\Psi_0'}$$

For example, suppose a Hamiltonian possesses a $\mathbb{Z}_n$ symmetry which is spontaneously broken. We can denote the symmetry operator associated to one of the n group elements as $\hat{S}$, where by the group law, composing the group element with itself n times yields the identity,

$$\hat{S}^n = \mathbb{1}$$

So, there will be n broken states, denoted

$$\hat{S}^\alpha \Psi_0\,, \qquad \alpha = 0, 1, \ldots, n-1$$

In general, for a finite group the order of the group will equal the number of degenerate ground states, and the ground states are related to one another by the action of the symmetry operator.

More generally, we could have the case that a symmetry group G (continuous or discrete) is partially broken, but preserves a subgroup $H \subset G$. So, G is the symmetry of the Lagrangian/Hamiltonian, and H is the symmetry of the ground state. This will lead to a degeneracy of G/H distinct states, where G/H denotes the group quotient (basically, what's leftover after modding out by the invariant subgroup H). For example, in the Heisenberg anti-ferromagnet, the $SU(2)$ rotational symmetry of the magnetic dipoles is broken to a $U(1)$ symmetry of the ground state that corresponds to rotations around a single axis. The quotient is $SU(2)/U(1) \simeq S^2$, which corresponds to the set of points on the surface of a sphere. These points indicate the possible directions of the magnetization of the ground states.

——— *End Lecture 25.*

6.3 The connection between phase transitions and symmetry

The importance of symmetry in the study of phase transitions cannot be overstated. To illustrate this relationship, we'll consider the example of the *Ising model*. We will consider this system as a classical model in the thermodynamic limit.

[24] This is the tricky part: to have SSB, we need ground states (eigenstates of the Hamiltonian with lowest energy) which are also eigenstates of the order parameter operator. But, the order parameter does not typically commute with the Hamiltonian, so how can this be? In precisely the thermodynamic limit, even though $[\hat{O}, \hat{H}] \neq 0$, the expectation value $\langle [\hat{O}, \hat{H}] \rangle_{\Psi_0} \to 0$, so that the symmetry broken states are truly eigenstates of the Hamiltonian.

The classical Ising model refers to a lattice of N sites in d-dimensions, with a spin on each site that can either be spin up ($s_i = +1$) or spin down ($s_i = -1$). The energy is,

$$E = -B\sum_i s_i - J\sum_{\langle ij\rangle} s_i s_j \,. \tag{6.5}$$

The spins interact only with their nearest neighbors (which is denoted by the $\langle ij\rangle$ in the sum), with the strength of the interaction measured by a coupling J. When $J > 0$, the neighboring spins prefer to be aligned (all up, or all down) since this lowers the energy – this is called *ferromagnetic* behavior. In general, we can consider applying an external magnetic field B to the system of spins, which makes the spins want to align up to lower the energy. The question we would like to ask is: what is the physics of this model at finite temperature T?

The probability of the system being found in a configuration of spins $\{s_i\}$ is given by the Boltzmann distribution,

$$P[s_i] = \frac{e^{-E[s_i]/(k_B T)}}{Z}\,, \qquad Z = \sum_{\{s_i\}} e^{-E[s_i]/(k_B T)}$$

where Z is the partition function – the sum of the Boltzmann factors over all states. The average spin of the configuration is what we call the *magnetization* M, which is computed as,

$$M = \frac{1}{N}\langle\sum_i s_i\rangle = \frac{k_B T}{N}\frac{\partial \log Z}{\partial B}$$

This is the order parameter. Generally, we'll consider the thermodynamic limit of this system, where $N \to \infty$ and $V \to \infty$ with N/V held fixed, and we'll be interested in the $B = 0$ case (where strictly speaking we should take the limit $B \to 0$ after $N \to \infty$).

Unfortunately, the partition function is only exactly calculable on a $d = 1$ dimensional lattice (where it is simple enough to be a homework problem), or a $d = 2$ dimensional lattice at $B = 0$ (with the much-more-complicated solution originally due to Onsager). In higher dimensions, there are no exact solutions. In such cases, the useful scheme for thinking about this system, as pioneered by Lev Landau, is called *mean field theory*. Basically, we first do the sum over all configurations of spins with fixed average magnetization, and then subsequently sum over all possible magnetizations, expressing the partition function in terms of the free energy $F(M)$ as,

$$Z = \sum_M \left(\sum_{\{s_i\} \text{ for fixed } M} e^{-E[s_i]/(k_B T)} \right) = \sum_M e^{-F(M)/(k_B T)}$$

This is a "course graining", since we re-express the energy and other thermodynamic variables in terms of the average magnetization M rather than the microscopic spins of the system. This also allows us to treat the magnetization as a uniform field rather than a discrete variable.

Equilibrium in thermodynamic systems relies on minimizing the free energy, so to find the equilibrium state (*i.e.* the stable phase of the system) we need to minimize F as a function of our order parameter, M. Actually, the useful quantity to compute is the intensive free energy per unit spin, $f(M) = F(M)/N$. As we will see, the behavior of the free energy depends on the temperature.

Taking $B = 0$ in (6.5), the microscopic definition of the system has a $\mathbb{Z}_2$ symmetry that corresponds to flipping all the spins at once: $s_i \to -s_i$ for all i. For this case, Landau used a series of approximations to compute,

$$f(M) \approx \text{constant} + \frac{1}{2}(T - T_c)M^2 + \frac{1}{12}TM^4 + \ldots \tag{6.6}$$

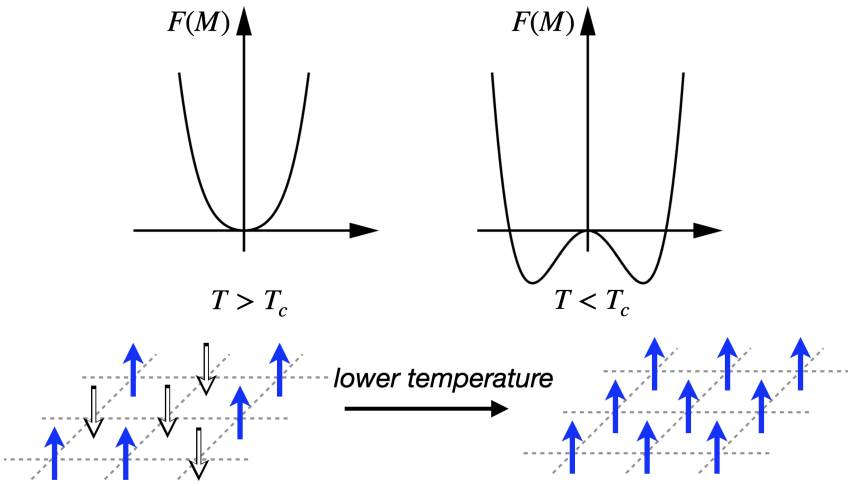


Figure 17: The phase transition in the Ising model is classified by spontaneous $\mathbb{Z}_2$ breaking.

where the "critical temperature" T_c depends on J and the number of dimensions – $T_c = 4J$ for a square lattice in 2 spatial dimensions, for instance. The crucial observation we can make about the Landau free energy (6.6) is that for $T > T_c$, the term in front of the M^2 is positive, and this is the anharmonic oscillator potential we already studied in (6.1). Much as we saw in that case, the ground state is given by expanding around the unique, symmetry-preserving minimum at $M = 0$. The conclusion is that at high temperatures $T > T_c$, the magnetization vanishes, $M = 0$, and the spins will be randomly distributed (with on average about half being pointed up and half being pointed down). This state preserves the underlying $\mathbb{Z}_2$ global spin-flip symmetry. This is known as the *disordered phase* of the system.

When $T < T_c$, however, the Landau free energy (6.1) takes the form of the spontaneous symmetry breaking double well potential we studied in (6.3). The minima now lie at $M = \pm M_0$, and the spins will either mostly aligned with either positive magnetization (up) or negative magnetization (down) depending on which ground state the system realizes. Even though the free energy is invariant under the $\mathbb{Z}_2$ symmetry (which acts on M as $M \to -M$), the lowest energy states break this symmetry by choosing either magnetization $+M_0$ or $-M_0$, which are related by the broken $\mathbb{Z}_2$. This is known as the *ordered phase*.

In this case, the transition at T_c is actually what's known as a *second order phase transition*, since the magnetization varies continuously as a function of T. When $B \neq 0$, on the other hand, the transition is first order (discontinuous). $B > 0$ tilts the free energy so that the minimum on the right is lower than the one on the left, and one minimum is preferred.

This example exhibits the general principles of symmetry breaking: when the underlying theory has a discrete symmetry, the ordered phase has a number of degenerate, disconnected ground states which spontaneously break the symmetry. In this case, $G = \mathbb{Z}_2$ was the underlying symmetry of the free energy. At high temperatures $T > T_c$, the symmetry was unbroken. At low temperatures, the symmetry was spontaneously broken so that the system must choose one of two ground states. The two different phases – disordered at high temperatures and ordered at low temperatures – are characterized by different choices of the symmetry which is preserved by the ground state: $H = \mathbb{Z}_2$ or $H =$ nothing. On the other hand, when $B \neq 0$ there is no $\mathbb{Z}_2$ symmetry of the free energy, and the two phases are not distinguishable. This is to say, by going to temperatures $T > T_c$ it is possible to move from any point in the phase diagram to any other point in the phase diagram without passing through a phase transition. This is precisely

the behavior of water in its transition between the liquid and gaseous phases near the critical temperature $T_c \approx 374^\circ C$. The similarity is not an accident: the critical point on the water phase diagram is well-described by the second order phase transition of the $d = 3$ Ising model! This is an example of the phenomenon known as *universality*: many different systems can be described by the same critical point.

This fits into the broader *Landau classification of phase transitions*: symmetry provides a powerful mechanism to understand when a phase transition will take place. We can characterize states of matter in terms of their (broken) symmetry. One determines an order parameter and a symmetry group G under which it transforms. Then, the different phases of matter within this class are characterized by the possible symmetry breaking patterns: the group H preserved by the ground state. The choice of universality class describing a second order phase transition is determined in large part by the symmetry breaking pattern.[25]

——— *End Lecture 26.*

[25] Warning: not all states of matter can be classified in this way! Understanding how generalized symmetry principles can extend this paradigm to encompass all known phases of matter is a fruitful area of modern research. For a nice review I suggest UCSD professor John McGreevy's take here, and public lecture available here.